Day 1 CAAM 567

Problems we will consider in this class

- Outliers
- Compressed Sensing
- Matrix Completion
- Robust PCA
- Phase Retrieval
- Blind Deconvolution
- Synchronization

Sparse

l_0

Notation: $[n] = \{1 .. n\}$

$\|\cdot\|_1$

support

convex funct

convex set

Activity

Suppose $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $x_0 = 0$.

Suppose you measure $x_i - x_j$ $\forall\, i,j$

$$x_3 - x_2 = 1 \qquad x_2 - x_1 = 1 \qquad x_1 - x_0 = 1$$

$$x_3 - x_1 = 2 \qquad x_2 - x_0 = 1$$

$$x_3 - x_0 = 1$$

What do you think x_0, x_1, x_2, x_3 are?

What principle would allow you to select these values?

Compressed Sensing

Let $x_0 \in \mathbb{R}^n$. $\|x_0\|_0 = k << n$.

Let $a_i \in \mathbb{R}^n$, $i = 1 \cdots m$ $(m < n)$

Let $b_i = a_i \cdot x_0$.

Given: $\{a_i\}$, b

Find: x_0

Basis Pursuit as a way to solve Compressed Sensing

Let: $X_0 \in \mathbb{R}^n$

$\|X_0\| = k << n$

$a_i \in \mathbb{R}^n \quad i = 1 \ldots m$

$m < n$

$b_i = a_i \cdot X_0$

Given: $\{a_i\}, b$

Find: X_0

Convex Program:

$$\min_{X \in \mathbb{R}^n} \|X\|_1 \quad \text{st} \quad \underbrace{b_i = a_i \cdot X, \; i = 1 \ldots m}_{b = \mathbf{A}x, \text{ where rows of } A \text{ are } a_i} \qquad (BP)$$

"basis pursuit"

Recovery Guarantee

If $a_i \sim N(0, I_n)$, $i = 1 \ldots m$ and

$m \geq C' k \log(\frac{n}{k})$

$\|X_0\| = k$

then with probability at least $1 - 2e^{-cm}$,

X_0 is the unique minimizer of (BP)

Why is this interesting?

If you knew which k entries were nonzero, it would take $m \geq k$ measurements to find x_0.

Even if you dont know support(x_0), only takes $\text{Const} \times k \times \log(\frac{n}{k})$ factor

the problem gets "worse" ~~harder~~ with larger dimension, but only logarithmically

Why probability?

Solving $\min_{x \in \mathbb{R}^n} \|x\|_0$ st $Ax = b$

is NP-hard.

Can only hope for recovery guarantee in a "typical" case. With random sensing vectors, there is always some probability they approx. align in a bad way. Hopefully that probability is small.

Why convex programming?

$$\min \underbrace{f(x)}_{\text{convex}} \quad \text{st} \quad x \in \underbrace{S}_{\text{convex}}$$

- No local minima.
  can descend down gradients.

Not convex

convex

- Often allows efficient algorithms that exist for
  linear program
  semidefinite programs

Tools:

- Least Squares & Linear Algebra
- Random numbers, vectors, matrices
- Convex optimization
- Nonsmooth functions

Matrix Completion

Let $M \in \mathbb{R}^{m \times n}$. Let Ω be a subset of entries.

Let $\mathrm{rank}(M) \ll \min(m,n)$

Given: M_Ω

Find: M.

Robust PCA

Let $M \in \mathbb{R}^{m \times n}$. Let $\Omega \subseteq [m] \times [n]$

Let $\operatorname{rank}(M) \ll \min(m,n)$

Let $E \in \mathbb{R}^{m \times n}$, $\|E\|_0 \ll mn$

Let $A = M_\Omega + E_\Omega$.

Given: A

Find: E

Synchronization.

~~Let $R_{ij} \in SO(2)$ for $(i,j) \in [n] \times [n]$~~

Let $R_i \in SO(2)$ for $i = 1 \cdots n$

Let $R_{ij} = \begin{cases} R_i R_j^{-1} & \text{for } (i,j) \in \Omega_g \subset [n] \times [n] \\ \text{arbitrary} & \text{for } (i,j) \in \Omega_b \subset [n] \times [n] \end{cases}$

Given: $\{R_{ij}\}$

Find: $\{R_i\}$

Outliers

Let $x_0 \in \mathbb{R}^n$

Let $a_i \in \mathbb{R}^n$ $i=1 \dots m$, $m \geq n$

Let ~~Given:~~ $b_i := a_i \cdot x_0$ for $i=1 \cdots m$,

Except some fraction are wrong

$b_i = a_i \cdot x_0 + \varepsilon_i$ where $\varepsilon \in \mathbb{R}^m$ is sparse

Given: $\{a_i\}, b$

Find: x_0