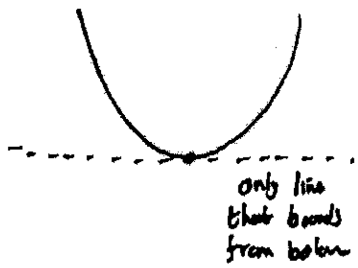
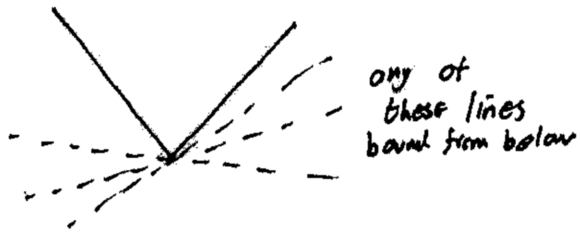
Subgradients (for convex functions)

Derivative/gradient notion for nonsmooth convex functions





v is <u>subgradient</u> of $f: \mathbb{R}^n \to \mathbb{R}$ at y if $f(x) - f(y) \geq \langle v, x-y \rangle \quad \forall x$

The subdifferential at y, $\partial f(y)$, is set of all subgradients at y. (set valued notion of derivative)

If f smooth $\quad \partial f(y) = \{ \nabla f(y) \}$

If $f(x) = |x|$, $\quad \partial f(0) = [-1, 1]$

If $f(x) = \|x\|_1$ for $x \in \mathbb{R}^n$, $\quad \partial f(0) = \{ x \mid \|x\|_\infty \leq 1 \}$

$$\partial f(y) = \left\{ x \;\middle|\; \begin{array}{ll} x_i = \operatorname{sign}(y_i) & \text{if } y_i \neq 0 \\ |x_i| \leq 1 & \text{if } y_i = 0 \end{array} \right.$$

Examples:

Let $I_S(x) = \begin{cases} 0 & \text{if } x \in S \\ \infty & \text{o'wise} \end{cases}$

Find $\partial I_{\{x\in\mathbb{R}^2 \mid x\geq 0\}}(x) = \begin{cases} \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\} & \text{if } x_1>0,\ x_2>0 \\ \left\{\begin{pmatrix}a\\0\end{pmatrix} \middle| a<0\right\} & \text{if } x_1=0,\ x_2>0 \\ \left\{\begin{pmatrix}0\\a\end{pmatrix} \middle| a<0\right\} & \text{if } x_1>0,\ x_2=0 \\ \left\{\begin{pmatrix}a\\b\end{pmatrix} \middle| a<0,\ b<0\right\} & \text{if } x_1=0,\ x_2=0 \\ \emptyset & \text{o'wise} \end{cases}$

~~Let~~

Find $\partial I_{\{x\in\mathbb{R}^2 \mid x_1+x_2=0\}}(x) = \begin{cases} \left\{c\begin{pmatrix}1\\1\end{pmatrix} \middle| c\in\mathbb{R}\right\} & \text{if } x_1+x_2=0 \\ \emptyset & \text{o'wise} \end{cases}$

Find $\partial I_{\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}}(x) = \begin{cases} \mathbb{R}^2 & \text{if } x=\begin{pmatrix}0\\0\end{pmatrix} \\ \emptyset & \text{o'wise} \end{cases}$

Is it true

$$\partial I_{\{x\in\mathbb{R}^2 \mid x\geq 0\}}\begin{pmatrix}0\\0\end{pmatrix} + \partial I_{\{x\in\mathbb{R}^2 \mid x_1+x_2=0\}}\begin{pmatrix}0\\0\end{pmatrix} = \partial I_{\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}}\begin{pmatrix}0\\0\end{pmatrix} \ ?$$

LHS $= \left\{\begin{pmatrix}a\\b\end{pmatrix} \middle| a<0, b<0\right\} + \left\{c\begin{pmatrix}1\\1\end{pmatrix} \middle| c\in\mathbb{R}\right\}$

RHS $= \mathbb{R}^2$

Yes, that are equal

Is the subgradient of a sum equal to sum of subgradients

$$\partial(f+g)_{(x)} = \partial f(x) + \partial g(x) \; ?$$

No.

Theorem: Moreau-Rockafeller

Let $f, g : \mathbb{R}^n \to (-\infty, \infty]$ be convex.

Let $x_0 \in \mathbb{R}^n$.

$$\partial f(x_0) + \partial g(x_0) \subset \partial(f+g)(x_0)$$

If $\operatorname{int} \operatorname{dom}(f) \cap \operatorname{dom} g \neq \phi$ then

$$\partial(f+g)(x_0) \subset \partial f(x_0) + \partial g(x_0)$$

Exercise: If x minimizes f, what does that mean in terms of subgradients?

Suppose f is convex, and let $h(x) = f(Ax+b)$
what is $\partial h(x)$?

$$A^t \partial f(Ax+b)$$