Convex Hull

Given $S=\{x\}\subset \mathbb{R}^n$, $\mathrm{conv}(S)$ is convex hull

$$\mathrm{conv}(S)=\{\theta_1 x_1+\dots+\theta_k x_k \mid x_i\in S,\ \theta_i\ge 0,\ \sum\theta_i=1\}$$

$\mathrm{conv}(S)$ is smallest convex set containing S.

Ex: $\{x\in\mathbb{R}^n \mid \|x\|_1\le 1\} = \mathrm{conv}\{e_1,-e_1,e_2,-e_2,\dots,e_n,-e_n\}$

Activity:

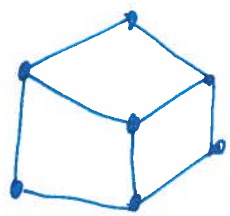
Find $\text{conv}(\ \{x \in \mathbb{R}^n \mid a_i \cdot x \leq 1 \text{ or } a_i \cdot x = -1\}$

Find $\text{conv}(\ \{x \in \mathbb{R}^2 \mid x_1 x_2 = 1\}\ )$

Extreme points of convex sets

If A is convex, $x \in A$ is an extreme point of A
if $x = \theta y + (1-\theta) z$ for $y, z \in A$ and $\theta \in [0,1] \Rightarrow y = x$ or $z = x$.

"x is not an interior point of any line segment in A"



← extreme points

whole boundary are extreme points

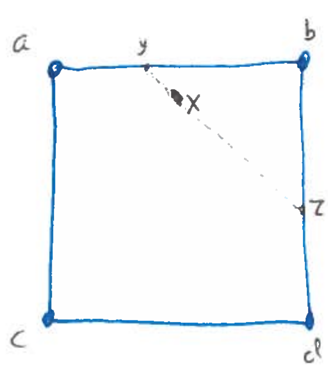
Theorem (Minkowski - Carathéodory)

Let A be a compact convex subset of $\mathbb{R}^n$ of dimension n
Then any $x \in A$ can be written as convex combination of at most $n+1$ extreme points.

dim of smallest affine space containing A

$$A = \text{conv}(\{\text{extreme points of } A\})$$

Example



x is conv combo of y & z
y & z are conv combos of a & b and b & d
x is conv combo of a, b, d.
$\mathbb{R}^2$. Need 3 points convex combos

Convex Program

$$\min f(x) \quad \text{st} \quad x \in S \qquad (*)$$

$(*)$ is a <u>convex program</u> if f is convex & S is convex.

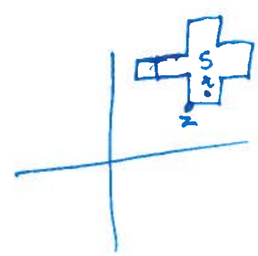
Convex Programs are nice b/c

- no local minimizers that arent global minimizers

function is
not convex

- you can find global minimizer by descending downhill from any point

Example $\min \|x\|_2$ st $x \in S$ for



is not a convex program

If you start at x and move downhill, you get stuck at Z, which is not minimizer.

Example:

Activity

Consider $\max_{x \in \mathbb{R}^2} \|x\|_2$ st $x \in S$ for a convex S. (*)

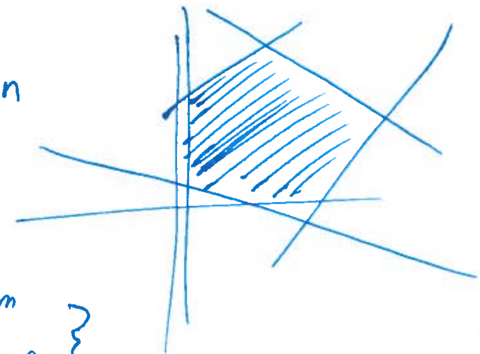
- Find an bounded convex S which has multiple local maximizers to (*)

- Find a bounded convex S which has only one local/global minimizer.

Linear Program

$$\min \; c^t x \quad \text{st} \quad a_i^t x \le b_i \quad i = 1 \dots m$$

Feasible set is a Polyhedron



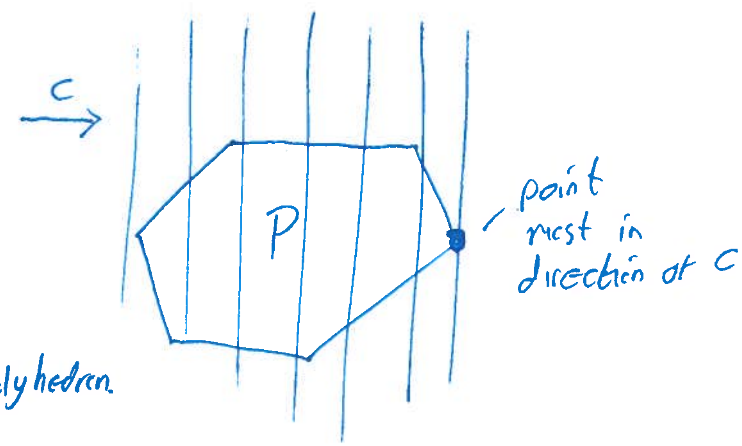
A polyhedron $P = \{x \mid a_i^t x \le b_i \;\; i=1\dots m, \;\; c_j^t x = d_j \;\; j = 1 \dots p\}$

is intersection of finite # halfspaces & hyperplanes.

Notation $\quad P = \{x \mid Ax \le b, \; Cx = d\}$

interpreted entrywise

Visual of (LP)



A polytope is a bounded polyhedron.

A polytope is convex hull of finite # of points (its vertices/extreme points)

For any LP over a polytope, a vertex is a minimizer (perhaps nonunique)

Some

NP-hard problems can be written as LPs

$d: V \times V \to \mathbb{R}_+$

Let $V = \{1, 2, \dots, n\}$. Let $d_{ij} = d_{ji}$

Let $\mathcal{F} = \{ (x_1, x_2, \dots, x_n) \mid (x_1 \cdots x_n) \text{ is hamiltonian cycle} \} \subseteq \underbrace{2^V}_{\text{all subsets of } V}$

$$\min \sum_{(i,j) \in e} d_{ij} \quad \text{subject to } e \in \mathcal{F}.$$

hamiltonian cycle

Find cycle of minimal total distance

To write as a LP, consider $\mathbb{R}^{\binom{n}{2}}$

Every hamiltonian cycle corresponds to an element of $\{0,1\}^{\binom{n}{2}}$

(Each edge is present "1" or not present "0".

$$\min \sum_{i \neq j} d_{ij} X_{(i,j)} \quad \text{s.t.} \quad X \in \{\text{Hamiltonian Cycles}\} \subset \{0,1\}^{\binom{n}{2}}$$

Convex relaxation: Replace feasible set by its conv. hull.

$$\min \sum_{i \neq j} d_{ij} X_{(i,j)} \quad \text{st} \quad X \in \text{Conv}(\{\text{Hamiltonian cycles}\})$$

Extreme points are Hamiltonian cycles. one (or more) will be global minimizer