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Analysis I
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Week 12 — Summary — Inverse Function Theorem

Reading: III.3, XVIII.1, XVIII.2, XVIII.3

131. A continuous, strictly increasing, real-valued function on $\mathbb{R}$ has an inverse that is continuous and strictly increasing.

132. A differentiable, strictly increasing function has an inverse that is differentiable and strictly increasing. The derivative of the inverse is the inverse of the derivative:

$$\frac{dy}{dx}(x) = \left(\frac{dx}{dy}(y)\right)^{-1}$$

133. Shrinking Lemma: Let M be a closed subset of a complete normed vector space. Let $f : M \to M$ be a mapping, and assume that there is a $0 < K < 1$ such that for all $x, y \in M$, $\|f(x) - f(y)\| \leq K\|x - y\|$. Then there exists a unique $x_0 \in M$ such that $f(x_0) = x_0$. If $x \in M$, then the sequence $\{f^n(x)\}$ coverages to x_0.

134. The set of invertible $n \times n$ matrices is open subset of all $n \times n$ matrices.

135. Let E be a complete normed vector space, and let $L(E, E)$ be the set of all linear maps from E to E. The set of invertible elements of $L(E, E)$ is open in $L(E, E)$. If $u \in L(E, E)$ is such that $\|u\| < 1$, then $I - u$ is invertible and $(I - u)^{-1} = \sum_{n=0}^{\infty} u^n$.

136. Let Inv(E, E) be the set of invertible elements of $L(E, E)$. Let $\phi : \text{Inv}(E, E) \to \text{Inv}(E, E)$ be the map $u \mapsto u^{-1}$. Then, ϕ is infinitely differentiable, and its derivative is given by $\phi'(u)v = -u^{-1}vu^{-1}$.

137. Let E, F be a complete normed vector spaces. Let U be open in E and let $f : U \to F$ be a C^p map. We say that f is C^p-invertible on U if the image of f is an open set V in F, and if there is a C^p map $g : V \to U$ such that $g(f(x)) = x$ and $f(g(y)) = y$ for all $x \in U$ and $y \in V$.

138. Inverse function theorem: Let U be open in E. Let $x_0 \in U$, and let $f : U \to F$ be a C^p map. Assume that the derivative $f'(x_0) : E \to F$ is invertible. The f is locally C^p-invertible at x_0. If ϕ is its local inverse, and $y = f(x)$, then $\phi'(y) = f'(x)^{-1}$.