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Analysis I
Paul E. Hand
hand@rice.edu

Week 11 — Summary — Differentiation of functions of multiple variables

Reading: XV.1-XV.2, XVII.1-XVII.3.
Let E, F, G be complete normed vector spaces.

119. Let U be an open set of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$. The ith partial derivative of f at $x \in U$ is

$$D_i f(x) = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}.$$

120. Theorem: Let $f : U \to R^n$. If $D_i f, D_j f, D_i D_j f, D_j D_i f$ exist and are all continuous on U, then

$$D_i D_j f = D_j D_i f \text{ on } U.$$

121. Relating partial derivatives in different coordinates. Let $x = r\cos\theta$, $y = r\sin\theta$. Let $f(x, y) = g(r, \theta)$. Then

$$\frac{\partial g}{\partial r} = D_1 f(x, y)\frac{\partial x}{\partial r} + D_2 f(x, y)\frac{\partial y}{\partial r}$$

and similarly for $\frac{\partial g}{\partial \theta}$.

122. We say that $f : U \to \mathbb{R}$ is differentiable at $x \in U$ if there exists an $A \in \mathbb{R}^n$ such that

$$f(x + h) = f(x) + A \cdot h + o(h).$$

Such an A is the derivative of f at x.

123. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $x \in U$ with derivative A. Then $A = \text{grad} f(x)$.

124. If all partial derivatives of $f : U \subset \mathbb{R}^n \to \mathbb{R}$ exist and are continuous in some open set containing x, then f is differentiable at x.

125. Let $\phi : [a, b] \to \mathbb{R}^n$ be differentiable and have values in an open set $U \subset \mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a differentiable function. The $f \circ \phi : J \to \mathbb{R}$ is differentiable and

$$(f \circ \phi)'(t) = \text{grad} f(\phi(t)) \cdot \phi'(t)$$

126. The direction of $\text{grad} f(x)$ is the direction of maximal increase of the function f at x.
The norm $|\text{grad} f(x)|$ is equal to the rate of change of f in its direction of maximal increase.
$\text{grad} f(x)$ is perpendicular to the level surface of f at x.

127. Any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by matrix multiplication for some matrix in $\mathbb{R}^{m \times n}$.

128. Let $U \subset E$. We say that $f : U \to F$ is differentiable at $x \in U$ if there exists a linear map $A : E \to F$ such that

$$f(x + h) = f(x) + Ah + o(h).$$

Such an A is the derivative of f at x, sometimes denoted by $f'(x)$.

129. Let U be an open subset of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$ be differentiable at x. The continuous linear map $f'(x)$ is represented by the Jacobian matrix

$$J_f(x) = \left(\frac{\partial f_i}{\partial x_j} \right).$$

130. Chain rule: Let U be open in E and let V be open in F. Let $f : U \to V$ and $g : V \to G$ be maps. Let $x \in U$. If f is differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x)) \circ f'(x)$$