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Analysis I
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Week 10 — Summary — Extensions of linear operators and the definition of integrals as limits of step functions

108. Definition: A linear operator (aka function or map) L from a normed vector space to another normed vector space is bounded if $\|L(x)\| \leq C\|x\|$ for all x. The constant C is an operator bound for L. The smallest such C is the operator norm of L.

109. A linear map from a normed vector space to another normed vector space is continuous if and only if it is bounded (as an operator).

110. Let F be a normed vector space, and let F_0 be a subspace. The closure of F_0 in F is a subspace of F.

111. Let F be a normed vector space, and let F_0 be a subspace. Let $L : F_0 \to E$ be a continuous linear map from F_0 into the complete normed vector space E. Then L has a unique extension to a continuous linear map $\bar{L} : \bar{F_0} \to E$ with the same operator bound.

112. A step function from $[a, b] \to E$, where E is a normed vector space, is a function of the form

$$f(x) = w_i \text{ for } a_{i-1} < t < a_i,$$

where $a = a_0 \leq a_1 \leq \ldots \leq a_n = b$ is a partition of $[a, b]$. Denote the set of step functions as $\text{St}([a, b], E)$.

113. The integral of a step function on $[a, b]$ is defined as $I(f) = \sum_{i=1}^{n} (a_i - a_{i-1}) w_i$.

114. $\text{St}([a, b], E)$ is a subspace of the space of all bounded maps from $[a, b]$ into E. The operator I is a linear operator from this subspace to E with bound $b - a$. That is, $\|I(f)\|_E \leq (b - a)\|f\|_\infty$.

115. The integral operator I can be extended to the closure of $\text{St}([a, b], E)$. We will call this closure the space of regulated maps, $\text{Reg}([a, b], E)$.

116. The closure of $\text{St}([a, b], E)$ contains $C^0([a, b], E)$. It also contains the class of piecewise continuous functions.

117. Let f be a regulated map on $[a, b]$. Let $F(x) = \int_a^x f(s)ds$. If f is continuous at the point c, then F is differentiable at c and $F'(c) = f(c)$.

118. Let $f(t, x)$ and $D_2 f(t, x)$ be defined and continuous for $(t, x) \in [a, b] \times [c, d]$. Then, for $x \in [c, d]$, $\frac{d}{dx} \int_a^b f(t, x)dt = \int_a^b D_2 f(t, x)dt$.