1 September 2016
Analysis I
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Week 3 — Summary — Riemann Integration

34. * A partition of $[a, b]$ is a sequence $(x_0, x_1, \ldots, x_n)$ where $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$. The size of the partition P is the length of its longest subinterval: $\|P\| = \max_i |x_{i+1} - x_i|$.

35. * The upper and lower Riemann sums of the function f under the partition P of $[a, b]$ are given by

$$U_a^b(f, P) = \sum_{i=0}^{n-1} M_i(f)(x_{i+1} - x_i)$$
$$L_a^b(f, P) = \sum_{i=0}^{n-1} m_i(f)(x_{i+1} - x_i)$$

where $M_i(f) = \sup_{x \in [x_i, x_{i+1}]} f(x)$ and $m_i(f) = \inf_{x \in [x_i, x_{i+1}]} f(x)$.

36. * If the infimum all upper sums of a function equals the supremum of all lower sums, we say the function is Riemann integrable. We say that the value of the integral is equal to the shared value of this infimum and supremum.

37. * All upper sums are at least as large as all lower sums. That is, for any partitions P_1, P_2 and function $f : [a, b] \to \mathbb{R}$,
$$U_a^b(f, P_1) \geq L_a^b(f, P_2)$$

38. *Darboux criterion: The function f is Riemann integrable on $[a, b]$ if and only if for all ε there is a partition P for which $U_a^b(f, P) - L_a^b(f, P) < \varepsilon$.

39. * Continuous functions are Riemann integrable (on closed bounded domains).

40. *The function f is Riemann integrable on $[a, b]$ with value s if and only if for all ε there is a δ such that $U_a^b(f, P) - s < \varepsilon$ and $s - L_a^b(f, P) < \varepsilon$ whenever $\|P_n\| < \delta$.

41. *The Riemann integral has several inadequacies.