23 August 2016
Analysis I
Paul E. Hand
hand@rice.edu

Week 1— Summary — Real Numbers, Limits and Continuous Functions

1. *Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the natural numbers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ be the integers.
2. *Let $\mathbb{Q}$ be the rationals. If $x \in \mathbb{Q}$, then $x = n/m$, for $n, m \in \mathbb{Z}$ and $m \neq 0$. There are a countable number of rationals.
3. *Let $\mathbb{R}$ be the reals. There are an uncountable number of reals. Each real number has a decimal representation (possibly two)
4. Some axioms of real numbers:
   (a) $(x + y) + z = x + (y + z)\ \forall x, y, z \in \mathbb{R}$ (additive associativity)
   (b) $0 + x = x + 0\ \forall x \in \mathbb{R}$ (additive identity)
   (c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 0$ (additive inverse)
   (d) $\forall x, y \in R, x + y = y + x$ (additive commutativity)
   (e) $(xy)z = x(yz)\ \forall x, y, z \in \mathbb{R}$ (multiplicative associativity)
   (f) $1x = x\ \forall x \in \mathbb{R}$ (multiplicative identity)
   (g) $\forall x \neq 0, \exists y$ such that $yx = 1$ (multiplicative inverse)
   (h) $xy = yx\ \forall x, y \in \mathbb{R}$ (multiplicative commutativity)
   (i) $x(y + z) = xy + xz\ \forall x, y, z \in \mathbb{R}$ (distributivity)
5. Completeness axiom of reals:
   (a) *Every non-empty set of reals which is bounded from above has a least upper bound. We denote the least upper bound of a set S by $\sup(S)$, which stands for the supremum of S. If S is unbounded from above, then we say that $\sup(S) = \infty$.
   (b) *Similarly, every non-empty set S which is bounded from below has a greatest lower bound, $\inf(S)$, which stands for the infimum of S. If S is unbounded from below, then we say that $\inf(S) = -\infty$.
6. Properties of the reals
   (a) Triangle inequality: For real numbers, $|x + y| \leq |x| + |y|$ and $|x - y| \geq |x| - |y|$.
   (b) Archimedian property: If $0 \leq x \leq 1/n\ \forall n \in \mathbb{N}$, then $x = 0$
   (c) Density of rationals within the reals: For all $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists $q \in \mathbb{Q}$ such that $|q - x| < \varepsilon$.
   (d) Between two distinct rationals, there is a real. Between two distinct reals, there is a rational.
7. *The sequence $\{x_n\}_{n=1}^{\infty}$ converges if $\exists\, a \in \mathbb{R}$ such that for all $\varepsilon > 0\ \exists N$ such that $n \geq N \Rightarrow |x_n - a| < \varepsilon$. We say that $\lim_{n \to \infty} x_n = a$.
8. *A bounded monotonic sequence converges.

9. *The sequence $\{x_n\}$ is Cauchy if $\forall\, \varepsilon > 0$, there exists N such that $m, n \geq N \Rightarrow |x_m - x_n| < \varepsilon$.

10. $\mathbb{R}$ is complete: If $\{x_n\}$ is a Cauchy sequence of $\mathbb{R}$, then $\{x_n\}$ converges to an element of $\mathbb{R}$.

11. Let $x = \{x_n\}$ be a sequence. A subsequence of x is obtained by keeping (in order) an infinite number of the items x_n and discarding the rest. Two ways to denote a subsequence are $x_{(n)}$ and x_{n_k}.

12. Let $\{x_n\}$ be a sequence. The number x is an accumulation point (or point of accumulation) of the sequence if $\forall \varepsilon$ there are infinitely many n such that $|x_n - x| < \varepsilon$.

13. *Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.

14. (a) *$\limsup\{x_n\}$ is defined as supremum of the accumulation points of $\{x_n\}$. An alternative way to think about it is through $\limsup\{x_n\} = \lim_{n->\infty} \sup_{m \geq n} x_m$.
    (b) *$\liminf\{x_n\}$ is defined analogously.

15. *Let f be a function defined on $S \subset \mathbb{R}$. The limit of $f(x)$ as x approaches a exists if there exists an L such that for all ε there is a $\delta > 0$ such that $|x - a| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$ for $x \in S$. We write such a limit as $\lim_{x \to a} f(x) = L$.

16. Limits commute with addition, multiplication, division, and non-strict inequalities
    (a) If $\lim_{x \to a}(cf)(x) = c \lim_{x \to a} f(x)$ for any real c.
    (b) If $\lim_{x \to a}(f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ if both limits on the right exist.
    (c) If $\lim_{x \to a}(fg)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ if both limits on the right exist.
    (d) If $\lim_{x \to a}(f/g)(x) = \lim_{x \to a} f(x)/\lim_{x \to a} g(x)$ if both limits on the right exist and the limit of g is nonzero.
    (e) If $f(x) \leq g(x)$ for all x sufficiently close to a, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$, provided both limits on the right exist.

17. The function $f : S \to \mathbb{R}$ is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

18. The function f is continuous on the set S if f is continuous at every point in S.

19. The composition of two continuous functions is continuous.

20. Intermediate value theorem: Let f be continuous on $[a, b]$. For any y satisfying $f(a) < y < f(b)$ or $f(b) < y < f(a)$, there exists an $x \in (a, b)$ such that $f(x) = y$.

21. *The function f is uniformly continuous on the set S if for all ε, there exists a $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Notice that the dependence of δ on ϵ does not depend on the position within the set. That is what makes it uniform.

22. *A continuous function on a closed, bounded interval is uniformly continuous.