3 September 2015
Analysis I
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HW 2

Due: 8 Sep 2015

The problems are written in the format 'chapter.section.problem-number' from Lang's book. Practice problems must be handed in and will be checked for honest effort. Portfolio problems will be graded thoroughly and may be revised until your solutions are of professional quality. Please submit each portfolio problem on a detached sheet of paper with your name on it.

Practice problems:

1. II.2.4
2. Let $g(x)$ be a bounded function in a neighborhood of a. Let $\lim_{x\to a} f(x) = 0$. Show that $\lim_{x\to a} f(x)g(x)$ exists and equals 0.
3. II.3.8
4. II.4.4
5. *The track problem.* Here is a claim: if the temperature of a circular running track is given by a continuous function of a single position variable, there are two diametrically opposite points that have equal temperature.
   (a) Write the claim as a formal statement about continuous functions.
   (b) Prove that the claim is true or find a counterexample. Hint: Think about the intermediate value theorem.

Portfolio problems:

P4. II.4.1

P5. Prove that a periodic continuous function on $\mathbb{R}$ is uniformly continuous or find a counterexample.

P6. In this problem you will find examples of functions $f_n(x)$ defined on $(0, 1)$ such that

$$\lim_{x\to 0} \sum_{n=1}^{\infty} f_n(x) \neq \sum_{n=1}^{\infty} \lim_{x\to 0} f_n(x).$$

(a) Find an example where the sum on the left hand side is $+\infty$ for all $x \in (0, 1)$.
(b) Find an example where all the sums and limits are finite.

(c) If $\{a_n\}$ and $\{b_n\}$ are two sequences of numbers, show that

$$\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$$

provided the limsups *on the right exist.*

Solution. The proofs of (a) and (b) are analoguous to the proofs given in the previous exercise.
(c) Let $\epsilon > 0$, $a = \limsup a_n$ and $b = \limsup b_n$. If $x \le a + \epsilon$ and $y \le b + \epsilon$, then $x + y \le a + b + 2\epsilon$. By (b) we conclude that there exists only finitely many n such that $a + b + 2\epsilon < a_n + b_n$. This implies that

$$\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n.$$

Exercise II.1.13 *Define the* **limit inferior** (lim inf). *State and prove the properties analoguous to those in Exercise 12.*

Solution. (a) Let S be a bounded set of real numbers. Let A be the set of its points of accumulation. Assume that A is non-empty. Then the greatest lower bound of A is called the **limit inferior** of S. If $b = \liminf S$, then b is also a point of accumulation of S because given $\epsilon > 0$ there exists a point of accumulation of S at distance $\epsilon/2$ hence there are infinitely many elements of S at distance $< \epsilon$ of b.
(b) We prove that a real number c is the limit inferior of S if and only if given $\epsilon > 0$ there exists only a finite number of x in S such that $x < c - \epsilon$ and there exists infinitely many x in S such that $x < c + \epsilon$. If $c = \liminf S$, then the second property holds because c is a point of accumulation of S and if the first property does not hold, then the Weierstrass-Bolzano theorem implies that there exists a point of accumulation b of S such that $b < c$ which is a contradiction. Conversely, suppose that c satisfies both properties. Then any ball of positive radius centered at c contains infinitely many points of S, so c is a point of accumulation of S. If there were a point of accumulation b of S with $b < c$, then the first property would be violated. So $c = \liminf S$, as was to be shown.

II.2 Functions and Limits

Exercise II.2.1 *Let $d > 1$. Prove: Given $B > 1$, there exists N such that if $n > N$, then $d^n > B$. [Hint: Write $d = 1 + b$ with $b > 0$. Then*

$$d^n = 1 + nb + \cdots \ge 1 + nb.]$$

Solution. Write $d = 1 + b$ with $b > 0$. By the binomial formula we get

$$d^n = (1+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^k = 1 + nb + \cdots \ge 1 + nb.$$

So given $B > 1$ choose N such that $N > (B-1)/b$. Then for all $n > N$, we have $d^n \ge 1 + nb > B$, as was to be shown.

Exercise II.2.2 *Prove that if $0 < c < 1$, then*

$$\lim_{n\to\infty} c^n = 0.$$

What if $-1 < c \le 0$? [Hint: Write $c = -1/d$ with $d > 1$.]

Solution. Write $c = 1/d$ with $d > 1$. Exercise 1 implies that given $\epsilon > 0$ there exists N so that for all $n > N$ we have $d^n > 1/\epsilon$. Then for all $n > N$, we get $c^n < \epsilon$. Hence $\lim_{n\to\infty} c^n = 0$.
If $c = 0$ the result is trivial. If $-1 < c < 0$, then $0 < |c| < 1$ and $\lim_{n\to\infty} |c|^n = 0$, so $\lim_{n\to\infty} c^n = 0$.

Exercise II.2.3 *Show that for any number $x \ne 1$ we have*

$$1 + x + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

If $|c| < 1$, show that

$$\lim_{n\to\infty} (1 + c + \cdots + c^n) = \frac{1}{1-c}.$$

Solution. We simply expand

$$\begin{aligned}(x-1)(x^n + x^{n-1} + \cdots + x + 1) &= x^{n+1} + x^n + \cdots + x - x^n - \cdots - x - 1 \\ &= x^{n+1} - 1.\end{aligned}$$

When $|c| < 1$, consider the difference

$$1 + c + \cdots + c^n - \frac{1}{1-c} = \frac{1 - c^{n+1} - 1}{1-c} = \frac{-c^{n+1}}{1-c}.$$

So

$$\left|1 + c + \cdots + c^n - \frac{1}{1-c}\right| \le \frac{|c|^{n+1}}{1-|c|}.$$

But $\lim_{n\to\infty} |c|^n = 0$, so the desired limit follows.

Exercise II.2.4 *Let a be a number. Let f be a function defined for all numbers $x < a$. Assume that when $x < y < a$ we have $f(x) \le f(y)$ and also that f is bounded from above. Prove that $\lim_{x\to a} f(x)$ exists.*

Solution. Let $a_n = a - 1/n$, defined for all large n and consider the sequence whose general term is given by $b_n = \{f(a_n)\}$. Then $\{b_n\}$ is an increasing sequence of real numbers and is bounded, so $\lim_{n\to\infty} b_n$ exists. Denote this limit by b. We contend that $\lim_{x\to a} f(x) = b$. Clearly, for each x, we have $f(x) \le b$ because there exists a_n (depending on x) such that

$x \le a_n$ and therefore $f(x) \le f(a_n) \le b$. Given $\epsilon > 0$ choose N such that $b - b^n < \epsilon$ whenever $n \ge N$. If $a_N \le x \le a$, then

$$b_N = f(a_N) \le f(x) \le b$$

so $0 \le b - f(x) \le b - b_N < \epsilon$, which proves our contention.

Exercise II.2.5 *Let $x > 0$. Assume that the n-th root $x^{1/n}$ exists for all positive integers n. Find $\lim_{n\to\infty} x^{1/n}$.*

Solution. If $x = 1$ the result is trivial. Suppose $x > 1$ and write $x^{1/n} = 1 + h_n$ with $h_n > 0$. Then

$$x = (1 + h_n)^n \ge 1 + nh_n.$$

This implies that

$$0 \le h_n \le \frac{x-1}{n}.$$

Therefore $\lim_{n\to\infty} h_n = 0$ and hence $\lim_{n\to\infty} x^{1/n} = 1$.

If $0 < x < 1$, then $1 < 1/x$ and $\lim_{n\to\infty}(1/x)^{1/n} = 1$. Hence $\lim_{n\to\infty} x^{1/n} = 1$.

Exercise II.2.6 *Let f be the function defined by*

$$f(x) = \lim_{n\to\infty} \frac{1}{1+n^2x}.$$

Show that f is the characteristic function of the set $\{0\}$, that is $f(0) = 1$ and $f(x) = 0$ if $x \ne 0$.

Solution. We have

$$f(0) = \lim_{n\to\infty} \frac{1}{1+0} = 1,$$

so $f(0) = 1$. If $x \ne 0$, choose N so that $N^2|x| - 1 > 1/\epsilon$. Then for all $n \ge N$ we have

$$\left|\frac{1}{1+n^2x}\right| \le \frac{1}{n^2|x|-1} < \epsilon,$$

so $f(x) = 0$.

II.3 Limits with Infinity

Exercise II.3.1 *Formulate completely the rules for limits of products, sums, and quotients when $L = -\infty$. Prove explicitly as many of these as are needed to make you feel comfortable with them.*

Solution. If M is a number > 0, then $\lim_{n\to\infty} f(x)g(x) = -\infty$ because given any $B > 0$ we can find numbers C_1 and C_2 such that for all $x > C_1$ we have $g(x) > M/2$ and such that for all $x > C_2$ we have $f(x) < -2B/M$. Then for all $x > \max(C_1, C_2)$ we have $B < -f(x)g(x)$.

If $M = \infty$, then $\lim_{n\to\infty} f(x)g(x) = -\infty$ because given any $B > 0$ we can find numbers C_1 and C_2 such that for all $x > C_1$ we have $g(x) > 1$ and such that for all $x > C_2$ we have $f(x) < -B$. Then $x > \max(C_1, C_2)$ implies $f(x)g(x) < -B$.

If M is a number, then $\lim_{n\to\infty} f(x) + g(x) = -\infty$. Choose C_1 such that $x > C_1$ implies $g(x) < M + 1$. Choose C_2 such that $x > C_2$ implies $f(x) < -B - M - 1$. Then for all $x > \max(C_1, C_2)$ we have $f(x) + g(x) < -B$.

If M is a number $\ne 0$, then $\lim_{n\to\infty} g(x)/f(x) = 0$. Indeed, there exists a number K (fixed) such that for all large x we have $|g(x)| < K$. Given ϵ there exists a number C_2 such that for all $x > C_2$ we have $|f(x)| > K/\epsilon$ and $|g(x)| < K$. Then for all $x > C_2$ we have $|g(x)/f(x)| < \epsilon$.

Exercise II.3.2 *Let $f(x) = a_dx^d + \cdots + a_0$ be a polynomial of degree d. Describe the behavior of $f(x)$ as $x \to \infty$ depending on whether $a_d > 0$ or $a_d < 0$. (Of course the case $a_d > 0$ has already been treated in the text.) Similarly, describe the behavior of $f(x)$ as $x \to -\infty$ depending on whether $a_d > 0$, $a_d < 0$, d is even, or d is odd.*

Solution. We can write

$$f(x) = a_dx^d\left(1 + \frac{a_{d-1}}{a_dx} + \cdots + \frac{a_0}{a_dx^d}\right)$$

for all large $|x|$. Since the expression in parentheses $\to 1$ as $|x| \to \infty$ we conclude that

$$\lim_{x\to\infty} f(x) = \infty$$

if $a_d > 0$ and

$$\lim_{x\to\infty} f(x) = -\infty$$

if $a_d < 0$.

Similarly we have

$$\lim_{x\to-\infty} f(x) = \infty$$

if $a_d > 0$ and d is even or $a_d < 0$ and d is odd. Also

$$\lim_{x\to-\infty} f(x) = -\infty$$

if $a_d > 0$ and d is odd or $a_d < 0$ and d is even.

Exercise II.3.3 *Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial. A* **root** *of f is a number c such that $f(c) = 0$. Show that any root satisfies the condition*

$$|c| \le 1 + |a_{n-1}| + \cdots + |a_0|.$$

[Hint: Consider $|c| \le 1$ and $|c| > 1$ separately.]

2) Fix ε.

By assumption on boldness of g $\exists\, \delta_0, M$ st $|g(x)| \leq M$ when $|x-a| \leq \delta_0$

By assumption that $\lim f = 0$, $\exists\, \delta_1$ such that $|f(x)| < \frac{\varepsilon}{M}$ when $|x-a| \leq \delta_1$

Choose $\delta = \min(\delta_0, \delta_1)$,

When $|x-a| \leq \delta$, $|g(x) f(x)| \leq M \cdot \frac{\varepsilon}{M} = \varepsilon$ ∎

Solution. If $|c| \le 1$, the result is trivial. Suppose $|c| > 1$. Since c is a root of f we have $-c^n = a_{n-1}c^{n-1} + \cdots + a_0$, thus

$$|c|^n \le |a_{n-1}||c|^{n-1} + \cdots + |a_0|.$$

Dividing by $|c|^{n-1}$ implies

$$|c| \le |a_{n-1}| + \cdots + \frac{|a_0|}{|c|^{n-1}},$$

but since $0 < 1/|c| < 1$, we get

$$|c| \le |a_{n-1}| + \cdots + |a_0| \le 1 + |a_{n-1}| + \cdots + |a_0|,$$

as was to be shown.

Exercise II.3.4 *Prove: Let f, g be functions defined for all sufficiently large numbers. Assume that there exists a number $c > 0$ such that $f(x) \ge c$ for all sufficiently large x, and that $g(x) \to \infty$ as $x \to \infty$. Show that $f(x)g(x) \to \infty$ as $x \to \infty$.*

Solution. There exists $C_1 > 0$ such that for all $x > C_1$, $f(x)$ is defined and $f(x) \ge c$. Let $B > 0$. There is a $C_2 > 0$ such that whenever $x > C_2$ we have $g(x) > B/c$. Then whenever $x > \max(C_1, C_2)$ we have $f(x)g(x) > B$ and therefore $f(x)g(x) \to \infty$ as $x \to \infty$.

Exercise II.3.5 *Give an example of two sequences $\{x_n\}$ and $\{y_n\}$ such that*

$$\lim_{n\to\infty} x_n = 0, \quad \lim_{n\to\infty} y_n = \infty,$$

and

$$\lim_{n\to\infty} (x_n y_n) = 1.$$

Solution. Take, for example, $x_n = 1/n$ and $y_n = n$ defined for $n \ge 1$.

Exercise II.3.6 *Give an example of two sequences $\{x_n\}$ and $\{y_n\}$ such that*

$$\lim_{n\to\infty} x_n = 0, \quad \lim_{n\to\infty} y_n = \infty,$$

and $\lim_{n\to\infty}(x_n y_n)$ does not exists, and such that $|x_n y_n|$ is bounded, i.e. there exists $C > 0$ such that $|x_n y_n| < C$ for all n.

Solution. Let $x_n = (-1)^n/n$ and $y_n = n$. Then $x_n y_n = (-1)^n$ and $|x_n y_n| \le 1$ for all $n \ge 1$.

Exercise II.3.7 *Let*

$$f(x) = a_n x^n + \cdots + a_0$$
$$g(x) = b_m x^m + \cdots + b_0$$

be polynomials, with $a_n, b_m \neq 0$, so of degree n, m respectively. Assume that $a_n, b_m > 0$. Investigate the limit

$$\lim_{n\to\infty} \frac{f(x)}{g(x)},$$

distinguishing the cases $n > m$, $n = m$, and $n < m$.

Solution. For large values of x we can write

$$\frac{f(x)}{g(x)} = \frac{a_n x^n \left(1 + \cdots + \frac{a_0}{a_n x^n}\right)}{b_m x^m \left(1 + \cdots + \frac{b_0}{b_m x^m}\right)},$$

where $(1 + \cdots + a_0/a_n x^n)$ and $(1 + \cdots + b_0/b_m x^m) \to 1$ as $x \to \infty$. Thus we have the three cases

$$\begin{cases} n > m \Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty, \\ n = m \Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = a_n/b_m, \\ n < m \Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0. \end{cases}$$

Exercise II.3.8 *Prove in detail: Let f be defined for all numbers $>$ some number a, let g be defined for all numbers $>$ some number b, and assume that $f(x) > b$ for all $x > a$. Suppose that*

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} g(x) = \infty.$$

Show that

$$\lim_{x\to\infty} g(f(x)) = \infty.$$

Solution. For all $x > a$, $g(f(x))$ is defined. Let $B > 0$. Choose $M_g > b$ such that for all $x > M_g$ we have $g(x) > B$. Choose $M_f > a$ such that for all $x > M_f$ we have $f(x) > M_g$. Then for all $x > M_f$ we have $g(f(x)) > B$.

Exercise II.3.9 *Prove: Let S be a set of numbers, and let a be adherent to S. Let f be defined on S and assume*

$$\lim_{x\to a} f(x) = \infty.$$

Let g be defined for all sufficiently large numbers, and assume

$$\lim_{x\to\infty} g(x) = L,$$

where L is a number. Show that

$$\lim_{x\to\infty} g(f(x)) = L.$$

Exercise II.4.3 *Let f be the function such that $f(x) = 0$ if x is irrational and $f(p/q) = 1/q$ if p/q is a rational number, $q > 0$, and the fraction is in reduced form. Show that f is continuous at irrational numbers and not continuous at rational numbers. [Hint: For a fixed denominator q, consider all fractions m/q. If x is irrational, such fractions must be at distance $> \delta$ from x. Why?]*

Solution. Suppose $x_0 = p_0/q_0$ is a rational number such that the fraction is in lowest form and $q_0 > 0$. Then $f(x_0) = 1/q_0$. Every non-trivial open interval contains an irrational number, therefore $f(x) = 0$ for x arbitrarily close to x_0. Thus f is not continuous at x_0, thereby proving that f is not continuous at rational numbers.

Let $\epsilon > 0$ and suppose x_0 is irrational. Let $q_0 \in \mathbf{Z}^+$ such that $1/q_0 < \epsilon$. For each $q \in \mathbf{Z}^+$ with $q \leq q_0$ let S_q be the set of $p \in \mathbf{Z}$ such that

$$\left|\frac{p}{q} - x_0\right| < 1.$$

The set S_q has finitely many elements. So there are only finitely many rationals with denominator $\leq q_0$ which are at distance less than 1 from x_0. So we can find δ such that all rationals in $(x_0 - \delta, x_0 + \delta)$ have denominator $> q_0$. To be precise we let

$$\operatorname{dist}(x_0, S_q) = \min_{p \in S_q}\left\{\operatorname{dist}\left(x_0, \frac{p}{q}\right)\right\}.$$

Then $\operatorname{dist}(x_0, S_q) > 0$ because x_0 is irrational, so select δ such that

$$0 < \delta < \min\{1, \min_{1 \leq q \leq q_0} \operatorname{dist}(x_0, S_q)\}.$$

Then $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

Exercise II.4.4 *Show that a polynomial of odd degree with real coefficients has a root.*

Solution. Suppose we have a polynomial

$$p(x) = a_m x^m + \cdots + a_0,$$

where m is odd and $a_m \neq 0$. We can assume without loss of generality that $a_m > 0$ (if not, consider $-p(x)$). Then we can write

$$p(x) = x^m\left[a_m + \frac{a_{m-1}}{x} + \cdots + \frac{a_0}{x^m}\right].$$

From this expression, it is clear that

$$\lim_{x\to\infty} p(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} p(x) = -\infty.$$

Since p is continuous, the intermediate value theorem implies that p has at least one real root.

Exercise II.4.5 *For $x \neq -1$ show that the following limit exists:*

$$f(x) = \lim_{n\to\infty}\left(\frac{x^n - 1}{x^n + 1}\right)^2.$$

(a) What are $f(1), f(\frac{1}{2}), f(2)$?
(b) What is $\lim_{x\to1} f(x)$?
(c) What is $\lim_{x\to-1} f(x)$?
(d) For which values of $x \neq -1$ is f continuous? Is is possible to define $f(-1)$ in such a way that f is continuous at -1.

Solution. If $x = 1$, then clearly, $f(x) = 0$. If $|x| > 1$, then

$$\left(\frac{x^n - 1}{x^n + 1}\right)^2 = \left(\frac{1 - 1/x^n}{1 + 1/x^n}\right)^2$$

so $f(x) = 1$. If $|x| < 1$, then

$$\lim_{n\to\infty}\left(\frac{x^n - 1}{x^n + 1}\right)^2 = \left(\frac{-1}{1}\right)^2 = 1.$$

(a) The above argument shows that $f(1) = 0, f(1/2) = 1$, and $f(2) = 1$.
(b) Note that $f(x) = 1$ for all x such that $|x| \neq 1$, but $f(1)$ is defined and $f(1) = 0$. So $\lim_{x\to1} f(x)$ does not exist, but

$$\lim_{x\to1, x\neq1} f(x) = 1.$$

(c) Similarly, $\lim_{x\to-1} f(x)$ does not exist, but

$$\lim_{x\to-1, x\neq-1} f(x) = 1.$$

(d) The function f is continuous at all $x \neq 1, -1$. However, f can be extended continuously as -1 by defining $f(-1) = 1$.

Exercise II.4.6 *Let*

$$f(x) = \lim_{n\to\infty}\frac{x^n}{1 + x^n}.$$

(a) What is the domain of definition of f, i.e. for which numbers x does the limit exist?
(b) Give explicitly the values $f(x)$ of f for the various x in the domain of f.
(c) For which x in the domain is f continuous at x?

Solution. (a) The domain of f is $\mathbf{R} - \{-1\}$.
(b) We have

$$\begin{cases} |x| < 1 \ \Rightarrow\ f(x) = 0, \\ |x| > 1 \ \Rightarrow\ f(x) = 1, \\ x = 1 \ \ \Rightarrow\ f(x) = 1/2. \end{cases}$$

(c) The function f is continuous for x such that $|x| < 1$ or $|x| > 1$.

5) The track problem

a) If $f \in C(\mathbb{R})$ and $f(x+L) = f(x)$ $\forall x$,

then $\exists$ x_0 such that $f(x_0) = f(x_0 + L/2)$

b) Proof: Let $g(x) = f(x + \frac{L}{2}) - f(x)$.

g is continuous.

Either $g(0) > 0$, $g(0) < 0$, or $g(0) = 0$.

- If $g(0) = 0$, we are done.
- If $g(0) > 0$, observe $g(\frac{L}{2}) = -g(0) < 0$.
By intermediate value theorem, $\exists$ x_0 s.t. $g(x_0) = 0$.
Hence $f(x_0) = f(x_0 + \frac{L}{2})$
- If $g(0) < 0$, same argument as $g(0) > 0$.

∎