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Analysis I
Paul E. Hand
hand@rice.edu

Day 7 — Summary — Riemann Integration

38. A partition of $[a, b]$ is a sequence $(x_0, x_1, \dots, x_n)$ where $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$. The size of the partition P is the length of its longest subinterval: $\|P\| = \max_i |x_{i+1} - x_i|$.

39. The upper and lower Riemann sums of the function f under the partition P of $[a, b]$ are given by

$$U_a^b(f, P) = \sum_{i=0}^{n-1} M_i(f)(x_{i+1} - x_i)$$
$$L_a^b(f, P) = \sum_{i=0}^{n-1} m_i(f)(x_{i+1} - x_i)$$

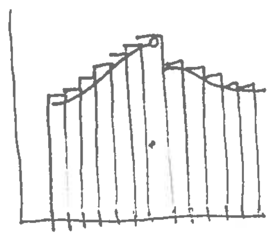
where $M_i(f) = \sup_{x \in [x_i, x_{i+1}]} f(x)$ and $m_i(f) = \inf_{x \in [x_i, x_{i+1}]} f(x)$.

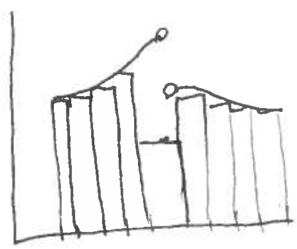
40. If the infimum all upper sums of a function equals the supremum of all lower sums, we say the function is Riemann integrable. We say that the value of the integral is equal to the shared value of this infimum and supremum.

41. All upper sums are at least as large as all lower sums. That is, for any partitions P_1, P_2 and function $f : [a, b] \to \mathbb{R}$,
$$U_a^b(f, P_1) \geq L_a^b(f, P_2)$$

38-40) Riemann Integral

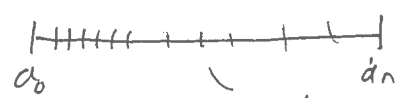
Visual:





Riemann integral exists if minimal area of upper sums (over any partition) equals max area of lower sums over any partition

Definitions: A partition P of $[a,b]$ is $(a_0, a_1, \dots a_n)$ such that
$$a = a_0 \le a_1 \le a_2 \le \dots \le a_n = b$$



Upper sum of a partition

$$U_a^b(f,P) = \sum_{i=0}^{n-1} \underbrace{M_i(f)}_{\substack{\sup f(x) \\ x \in [a_i, a_{i+1}]}} (a_{i+1} - a_i)$$

$$L_a^b(f,P) = \sum_{i=0}^{n-1} \underbrace{m_i(f)}_{\substack{\inf f(x) \\ x \in [a_i, a_{i+1}]}} (a_{i+1} - a_i)$$

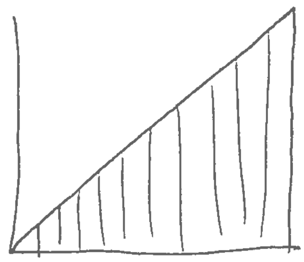
Riemann integral $\int_a^b f(x)dx = \sup_P L_a^b(f,P) = \inf_P U_a^b(f,P)$

if this sup = this inf.

Note automatically $\sup_P L_a^b(f,P) \leq \inf U_a^b(f,P)$

Example: $f(x) = x$ is Riemann integrable

Consider partition $P_n = (0, \frac{1}{n}, \frac{2}{n}, \dots, 1)$



$$L_0^1(f, P_n) = \sum_{i=0}^{n-1} \frac{1}{n}\frac{i}{n} = \frac{1}{n^2}\sum_{i=0}^{n-1} i = \frac{1}{n^2}\frac{n(n-1)}{2} = \frac{1}{2} - \frac{1}{2n}$$

So $\sup\limits_{P} L_0^1(f, P) \geq \frac{1}{2}$

$$U_0^1(f, P_n) = \sum_{i=0}^{n-1} \frac{1}{n}\frac{i+1}{n} = \sum_{i=1}^{n} \frac{1}{n}\frac{i}{n} = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n}$$

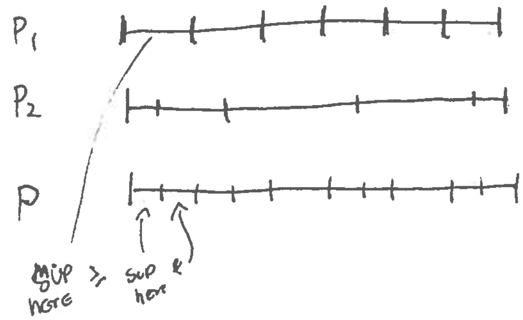
So $\inf\limits_{P} U_0^1(f, P) \leq 1/2$.

So $\sup\limits_{P} L_0^1(f, P) = \inf\limits_{P} U_0^1(f, P) = \frac{1}{2}$. Riemann integral $= 1/2$

4[I]) $\forall$ partitions P_1 & P_2, $\forall f : [a,b] \to \mathbb{R}$

$$U_a^b(f, P_1) \geq L_a^b(f, P_2)$$

Proof (gist)



Combine partits

So $U_a^b(f, P_1) \geq U_a^b(f, P) \geq L_a^b(f, P) \geq L_a^b(f, P_2)$.