10 September 2015
Analysis I
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## Day 6 — Summary — Convexity, Inverse Function Theorem

34. A function is convex if for all $t \in (0,1)$ and for all points $a$ and $b$,

$$f\Big((1-t)a + tb\Big) \leq (1-t)f(a) + tf(b).$$

It is strictly convex if this inequality is strict.

35. If $f''(x) > 0$ in an interval, then $f$ is strictly convex in the interval.

36. A continuous, strictly increasing function has an inverse that is continuous and strictly increasing.

37. A differentiable, strictly increasing function has an inverse that is differentiable and strictly increasing. The derivative of the inverse is the inverse of the derivative:

$$\frac{dy}{dx}(x) = \left(\frac{dx}{dy}(y)\right)^{-1}$$

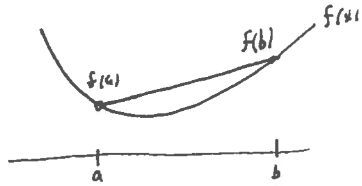
Exercise:

Find a function $f \in C^\infty(\mathbb{R})$ such that $f \equiv 0$ outside of $[-1,1]$.

Draw it first.
Put it in form $f(x) = e^{g(x)}$
Try to squish $e^{-x^2}$ into something finite.

34) ~~[scribbled out]~~ )

$f$ is convex if it is below its secant line segments




$$f(\underbrace{(1-t)a+tb}_{\substack{\text{convex}\\\text{combination}\\\text{of } a \,\&\, b}}) \leq \underbrace{(1-t)f(a)+tf(b)}_{\substack{\text{convex combination}\\\text{of } f(a) \,\&\, f(b)}}$$

Strictly convex if strict inequality

Examples:

Convex on $\mathbb{R}$ $\qquad f(x) = ax+b$

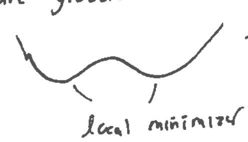
Not strictly convex on $\mathbb{R}$ $\qquad f(x) = |x|$

Strictly convex $\qquad f(x) = x^2$

Applications: - Convex optimization $\min f(x)$ is "Easy" if $f$ is convex. Even if $f$ isnt sm

- Note: all local minimizers are global

Cant have:



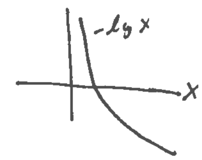

- not convex

-

Application: Arithmetic-geometric mean

$$\frac{a+b}{2} \geq \sqrt{ab} \quad \text{for } a,b \geq 0$$

why? $-\log(x)$ is convex




$$-\log\left(\frac{a+b}{2}\right) \leq \frac{-\log a - \log b}{2} \quad \text{by convexity.}$$

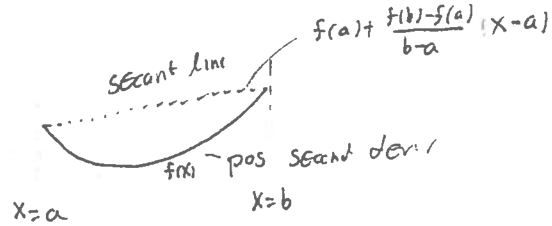
$$\log\left(\frac{a+b}{2}\right) \geq \frac{\log a + \log b}{2}$$

$$\frac{a+b}{2} \geq a^{1/2}\, b^{1/2} \quad \blacksquare$$

Application: Jensens inequality in probability.

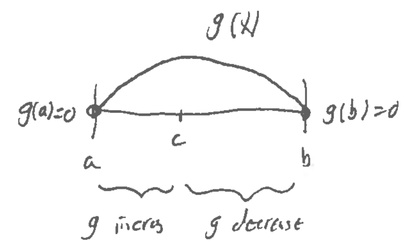
35) Theorem: If $f'' > 0$ on $[a,b]$, then $f$ convex on $[a,b]$

Visually:




Proof: Consider difference of secant line & $f$

$$g(x) := f(a) + \frac{f(b)-f(a)}{b-a}(x-a) - f(x)$$




Show $g' > 0$ for $x \in (a,c)$
$g' < 0$ for $x \in (c,b)$.

Compute, $g'(x) = \frac{f(b)-f(a)}{b-a} - f'(x)$

$$g'(x) = f'(c) - f'(x)$$
$$= f''(d)(c-x)$$

$> 0$ for $x > c$
$< 0$ for $x < c$

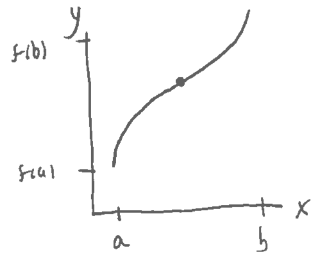
35)

Example of function that is strictly convex, yet $f''$ isnt always positive.

$f(x) = x^4$.

3b) Theorem: If $f \in C[a,b]$ is strictly increasing, then $f^{-1}$ exists and is continuous $C[f(a), f(b)]$ and strictly increasing

Visual:

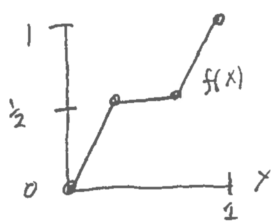



Meaning of $f^{-1}$:

$f^{-1}(y) = x$ such that $f(x) = y$.

For $f^{-1}$ to be well defined, there has to be exactly one $x$ such that $f(x) = y$ $\forall\ y \in [f(a), f(b$

Nonexample:




$f(x)$ is increasing but not strictly.

$f^{-1}(y)$ is not well-defined because $\exists$ many $x$ such that $f(x) = \frac{1}{2}$.

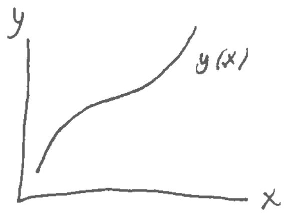

$f^{-1}(y)$ is not a function

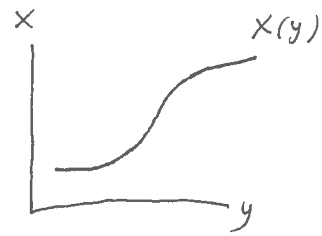
## 37) Inverse function Theorem

Idea: A diffable function that has ~~strictly increasing~~ positive derivative over a region has a diffable inverse ~~that has~~ and deriv of inverse is inverse of derivatives

Precise: If $y(x)$ diffable on $[a,b]$, $y'(x) > 0$ $\forall\, x \in [a,b]$
then $\exists$ $x(y)$ diffable on $[y(a), y(b)]$ with $\frac{dx}{dy}(y) = \frac{1}{\frac{dy}{dx}(x)} = \frac{1}{\frac{dy}{dx}(x(y))}$

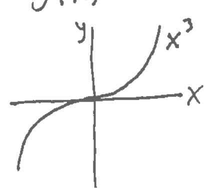
Visual:

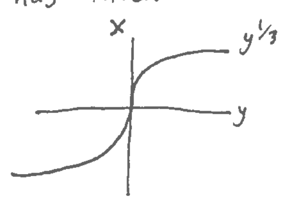



Why doesnt theorem assume that function is merely increasing?
Such a function has a continuous inverse but nothing but not a diffable one.

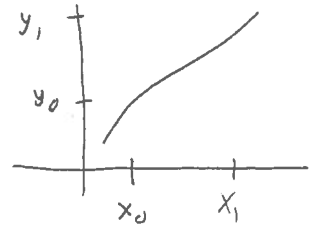
$y(x) = x^3$ has inverse $x(y) = y^{1/3}$ not diffable at $y=0$



Proof:

Let $y_0 = y(x_0)$
$y_1 = y(x_1)$




Note: $$\frac{x(y_1) - x(y_0)}{y_1 - y_0} = \frac{x_1 - x_0}{y(x_1) - y(x_0)} = \frac{1}{\frac{y(x_1) - y(x_0)}{x_1 - x_0}}$$

Because $x(y)$ & $y(x)$ is continuous,

$$x'(y_0) = \lim_{y_1 \to y_0} \frac{x(y_1) - x(y_0)}{y_1 - y_0} = \lim_{x_1 \to x_0} \frac{x_1 - x_0}{y(x_1) - y(x_0)} = \frac{1}{y'(x_0)}$$