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Analysis I
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Day 3 — Summary — Limits and continuity of functions

15. Let f be a function defined on $S \subset \mathbb{R}$. The limit of $f(x)$ as x approaches a exists if there exists an L such that for all ε there is a $\delta > 0$ such that $|x-a| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$ for $x \in S$. We write such a limit as $\lim_{x\to a} f(x) = L$.

16. Limits commute with addition, multiplication, division, and non-strict inequalities

    (a) If $\lim_{x\to a}(cf)(x) = c\lim_{x\to a} f(x)$ for any real c.

    (b) If $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ if both limits on the right exist.

    (c) If $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ if both limits on the right exist.

    (d) If $\lim_{x\to a}(f/g)(x) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$ if both limits on the right exist and the limit of g is nonzero.

    (e) If $f(x) \leq g(x)$ for all x sufficiently close to a, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$, provided both limits on the right exist.

17. The function $f : S \to \mathbb{R}$ is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

18. The function f is continuous on the set S if f is continuous at every point in S.

19. The composition of two continuous functions is continuous.

20. Intermediate value theorem: Let f be continuous on $[a, b]$. For any y satisfying $f(a) < y < f(b)$ or $f(b) < y < f(a)$, there exists an $x \in (a, b)$ such that $f(x) = y$.

21. The function f is uniformly continuous on the set S if for all ε, there exists a $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Notice that the dependence of δ on ϵ does not depend on the position within the set. That is what makes it uniform.

22. A continuous function on a closed, bounded interval is uniformly continuous.