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Analysis I
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Day 1— Summary — Real Numbers

1. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the natural numbers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ be the integers.
2. Let $\mathbb{Q}$ be the rationals. If $x \in \mathbb{Q}$, then $x = n/m$, for $n, m \in \mathbb{Z}$ and $m \neq 0$. There are a countable number of rationals.
3. Let $\mathbb{R}$ be the reals. There are an uncountable number of reals. Each real number has a decimal representation (possibly two)
4. Some axioms of real numbers:
   (a) $(x + y) + z = x + (y + z)\ \forall x, y, z \in \mathbb{R}$ (additive associativity)
   (b) $0 + x = x + 0\ \forall x \in \mathbb{R}$ (additive identity)
   (c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 0$ (additive inverse)
   (d) $\forall x, y \in R, x + y = y + x$ (additive commutativity)
   (e) $(xy)z = x(yz)\ \forall x, y, z \in \mathbb{R}$ (multiplicative associativity)
   (f) $1x = x\ \forall x \in \mathbb{R}$ (multiplicative identity)
   (g) $\forall x \neq 0, \exists y$ such that $yx = 1$ (multiplicative inverse)
   (h) $xy = yx\ \forall x, y \in \mathbb{R}$ (multiplicative commutativity)
   (i) $x(y + z) = xy + xz\ \forall x, y, z \in \mathbb{R}$ (distributivity)
5. Completeness axiom of reals:
   (a) Every non-empty set of reals which is bounded from above has a least upper bound. We denote the least upper bound of a set S by $\sup(S)$, which stands for the supremum of S. If S is unbounded from above, then we say that $\sup(S) = \infty$.
   (b) Similarly, every non-empty set S which is bounded from below has a greatest lower bound, $\inf(S)$, which stands for the infimum of S. If S is unbounded from below, then we say that $\inf(S) = -\infty$.
6. Properties of the reals
   (a) Triangle inequality: For real numbers, $|x + y| \leq |x| + |y|$ and $|x - y| \geq |x| - |y|$.
   (b) Archimedian property: If $0 \leq x \leq 1/n\ \forall n \in \mathbb{N}$, then $x = 0$
   (c) Density of rationals within the reals: For all $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists $q \in \mathbb{Q}$ such that $|q - x| < \varepsilon$.
   (d) Between two distinct rationals, there is a real. Between two distinct reals, there is a rational.
7. The sequence $\{x_n\}_{n=1}^{\infty}$ converges if $\exists\ a \in \mathbb{R}$ such that for all $\varepsilon > 0\ \exists N$ such that $n \geq N \Rightarrow |x_n - a| < \varepsilon$. We say that $\lim_{n\to\infty} x_n = a$.
8. A bounded monotonic sequence converges.