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Analysis I
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Day 19 — Summary — Series

108. If $\{x_n\}$ is a sequence in a normed vector space, we define the infinite sum $\sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n$. The infinite series converges if this sum exists. We say that an infinite series diverges if the partial sums are unbounded.

109. Comparison test. Let $\sum a_n$ and $\sum b_n$ be series of real numbers. If $\sum b_n$ converges and $0 \leq a_n \leq b_n$ for sufficiently large n, then $\sum a_n$ converges.

110. Ratio test. Let $\sum a_n$ be a series of nonnegative real numbers, and let $0 < c < 1$ be such that $a_{n+1} \leq ca_n$ for sufficiently large n. Then $\sum a_n$ converges.

111. Integral test. Let f be a decreasing function over all real numbers ≥ 1. The infinite series $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_a^{\infty} f(x)dx$ exists and is finite. Note that $\int_a^{\infty} f(x)dx$ is defined as $\lim_{M\to\infty} \int_1^M f(x)dx$.

112. Let $\sum a_n$ be a series of numbers. If $\sum |a_n|$ converges, then $\sum a_n$ converges. The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

113. Let $\{a_n\}$ be a sequence of numbers monotonically decreasing to zero. The alternating series $\sum (-1)^n a_n$ converges.

114. Let $\sum a_n$ be a series of vectors in a complete normed vector space. If $\sum \|a_n\|$ converges, then $\sum a_n$ converges. The series $\sum a_n$ is said to converge absolutely if $\sum \|a_n\|$ converges.

115. Let $\sum x_n$ be an absolutely convergent series in a complete normed vector space. Then the series obtained by any rearrangement of the series also converges absolutely to the same limit.

116. We say that an infinite series of functions $\sum_n f_n(x)$ converges absolutely on S if $\sum |f_n(x)|$ converges for all $x \in S$. We say the infinite series converges uniformly on S if the sequence of partial sums converges uniformly on S.

117. Weierstrass test: Let $f_n \in L^{\infty}$ be such that $\|f_n\|_{\infty} \leq M_n$ and $\sum M_n$ converges. Then $\sum f_n$ converges uniformly and absolutely. If each f_n is continuous, then so is $\sum f_n$.