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Analysis I
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Day 19 — Summary — Series

108. If $\{x_n\}$ is a sequence in a normed vector space, we define the infinite sum $\sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n$. The infinite series converges if this sum exists. We say that an infinite series diverges if the partial sums are unbounded.

109. Comparison test. Let $\sum a_n$ and $\sum b_n$ be series of real numbers. If $\sum b_n$ converges and $0 \leq a_n \leq b_n$ for sufficiently large n, then $\sum a_n$ converges.

110. Ratio test. Let $\sum a_n$ be a series of nonnegative real numbers, and let $0 < c < 1$ be such that $a_{n+1} \leq c a_n$ for sufficiently large n. Then $\sum a_n$ converges.

111. Integral test. Let f be a decreasing function over all real numbers ≥ 1. The infinite series $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_a^\infty f(x)dx$ exists and is finite. Note that $\int_a^\infty f(x)dx$ is defined as $\lim_{M\to\infty} \int_1^M f(x)dx$.

112. Let $\sum a_n$ be a series of numbers. If $\sum |a_n|$ converges, then $\sum a_n$ converges. The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

113. Let $\{a_n\}$ be a sequence of numbers monotonically decreasing to zero. The alternating series $\sum(-1)^n a_n$ converges.

114. Let $\sum a_n$ be a series of vectors in a complete normed vector space. If $\sum \|a_n\|$ converges, then $\sum a_n$ converges. The series $\sum a_n$ is said to converge absolutely if $\sum \|a_n\|$ converges.

115. Let $\sum x_n$ be an absolutely convergent series in a complete normed vector space. Then the series obtained by any rearrangement of the series also converges absolutely to the same limit.

116. We say that an infinite series of functions $\sum_n f_n(x)$ converges absolutely on S if $\sum |f_n(x)|$ converges for all $x \in S$. We say the infinite series converges uniformly on S if the sequence of partial sums converges uniformly on S.

117. Weierstrass test: Let $f_n \in L^\infty$ be such that $\|f_n\|_\infty \leq M_n$ and $\sum M_n$ converges. Then $\sum f_n$ converges uniformly and absolutely. If each f_n is continuous, then so is $\sum f_n$.

Warmup:

Provide a Cont function f over ℓ^∞ and a closed S. such that $\max f$ is finite & not achieved

10)

Converges: $\sum_{n=1}^{\infty} a^n$ for $|a|<1$ (direct computation)

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p>1$ (~~integral~~ test) ~~ratios~~ integral

Diverges: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p \leq 1$

$\sum_{n\geq 1}^{\infty} a^n$ for $a \in (-\infty, 1) \cup [1, \infty)$

Example: let $S_N = \sum_{i=0}^{N} a^n$

$$S_N = 1 + a + \cdots + a^N$$

$$aS_N = \quad a + \cdots + a^N + a^{N+1}$$

$$(a-1)S_N = a^{N+1} - 1$$

$$S_N = \frac{1-a^{N+1}}{1-a}$$

$$\lim_{N\to\infty} S_N = \frac{1}{1-a} \quad \text{if } |a|<1$$

10.4)

Let a_n, b_n ~~$\geq$~~ $0 \leq a_n \leq b_n$ for $n \geq N$.

Then $\sum_{n\geq 1}^{\infty} b_n$ converges $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

$\sum_{n\geq 1}^{\infty} a_n$ diverges $\Longrightarrow$ $\sum b_n$ diverges

Example:

$$\sum_{n\geq 1}^{\infty} \frac{\log n}{n} \text{ diverges by comparison to } \sum_{n=1}^{\infty} \frac{1}{n}.$$

Example: $\sum_{n\geq 1}^{\infty} \frac{\log^3 n}{n^2}$ converges, by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$

Because $\lim_{n\to\infty} \frac{\log^3 n}{\sqrt{n}} = 0$, $\exists N$ st $\log^3 n \leq \sqrt{N}$

Example: $\sum_{n=1}^{\infty} n^3 e^{-n}$ converges by comp to $\sum_{n=1}^{\infty} e^{-n/2}$

11c)   If $0 \leq a_{n+1} \leq c\, a_n$ for $c \in (0,1)$ and $n \geq N$,
then $\sum_{n=1}^{\infty} a_n$ converges.

Pf: Compare to $\sum_{n=1}^{\infty} c^n$ $\left(\text{or} \quad a_N \sum_{n=1}^{\infty} c^{(n-N)} \right)$

Example: $\cancel{\sum_{n=1}^{\infty} \frac{1}{n^2-1}}$

$\cancel{\sum_{n=1}^{\infty} \frac{1}{n^2}}$ ~~Note~~ $\cancel{a_n = \frac{1}{n^2}}$

$\cancel{\frac{a_{n+1}}{a_n} =}$

$\sum_{n=1}^{\infty} \frac{b^n}{n!}$ converges $a_n = \frac{b^n}{n!}$ $\frac{a_{n+1}}{a_n} = \frac{b^{n+1}}{(n+1)!} \frac{n!}{b^n} = \frac{b}{n+1} \to 0$

Ratio test applies for $c = 1/2$, eg.

112 $\cancel{}$) Let $a_n \in \mathbb{R}$ $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges $\Longrightarrow$ $\displaystyle\sum_{n=1}^{\infty} a_n$ converges

Idea: worst case of sum is when everything adds (no cancellation)

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$$

Proof: Let $S_N = \displaystyle\sum_{n=1}^{N} a_n$ Let $T_N = \displaystyle\sum_{n=1}^{N} |a_n|$

$$|S_N| = \left| \sum_{n=1}^{N} a_n \right| \leq \sum_{n=1}^{N} |a_n| = T_N \text{ bounded.}$$

So S_N is bdd and $\uparrow$. So S_N conv.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ conv bc $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv

Warm up

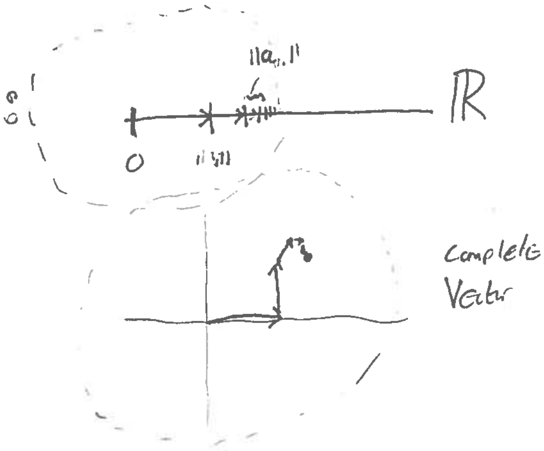
Converge or diverge

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

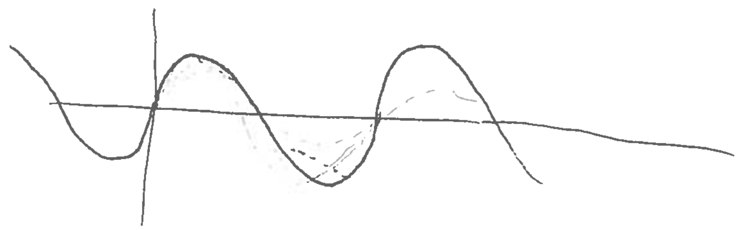
$$\sum_{n=1}^{\infty} \frac{1}{n \log n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n \log^2 n}$$

114) Visually:



Eg $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges in L^∞. to a continuous function (with ∞ slope at $x=0$



Thm: If $\sum_{n=1}^{\infty} \|a_n\| < \infty$ then $\sum_{n=1}^{\infty} a_n$ conv (in a complete normed v space)

Pf: Let $S_N = \sum_{k=1}^{n} a_n$ $\quad \|S_n - S_m\| = \|\sum_{k=n+1}^{m} a_n\| \le \sum \|a_n\|$

As $\sum_{n=1}^{\infty} \|a_n\| < \infty$, $\sum_{k=1}^{n} \|a_n\|$ Cauchy, hence $\forall \varepsilon \exists N$ st $n,m \ge N \Rightarrow \|S_n - S_m\| < \varepsilon$.

So S_N Cauchy complete, so S_N converges.

116 ~~#~~)

Does $\sum_{n=0}^{\infty} x^n$ converge uniformly on $|x|<1$?

No

Does $\sum_{n=0}^{\infty} x^n$ converge uniform on $|x| < 1-\varepsilon$?

Yes

$\sum_{n\geq 1}^{\infty} x^n$

$\|f^n\| \leq (1-\varepsilon)^n \qquad \sum_{n\geq 0}^{\infty} (1-\varepsilon)^n < \infty$