1 October 2015
Analysis I
Paul E. Hand
hand@rice.edu

Day 10 — Summary — Norms and Inner Products

51. A vector space V over the reals is a set that permits addition and scalar multiplication.
    (a) $(x + y) + z = x + (y + z)\ \forall x, y, z \in V$
    (b) $0 + x = x\ \forall x \in V$
    (c) $\forall x \in V, \exists y \in V$ such that $x + y = 0$
    (d) $x + y = y + x\ \forall x, y \in V$
    (e) For $x \in V$ and $a, b \in \mathbb{R}$, $(ab)x = a(bx), (a + b)x = ax + bx, a(x + y) = ax + ay$.
52. A norm on a vector space V is denoted by $\|\cdot\|$ and satisfies
    (a) $\|x\| \geq 0$ for all $x \in V$
    (b) $\|x\| = 0 \Leftrightarrow x = 0$.
    (c) $\|ax\| = |a|\|x\|$ for all $x \in V, a \in \mathbb{R}$
    (d) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$
53. For finite and infinite sequences x, the ℓ_p norm is $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$. It is a norm for $1 \leq p < \infty$. The ℓ_∞ or sup norm of a sequence x is $\|x\|_\infty = \sup_i |x_i|$.
54. For functions $f : \Omega \to \mathbb{R}$, the L_p norm is $\|f\|_p = \left(\int_\Omega |f|^p\right)^{1/p}$. The L_∞ norm is $\|f\|_\infty = \sup_{x\in\Omega} |f(x)|$.
55. A norm for $C^p[a, b]$ is given by $\|f\| = \sum_{i=0}^{p} \|f^{(i)}\|_\infty$.
56. Norms can be visualized by their unit ball.