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Analysis I
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Day 8 — Summary — Riemann Integration and Taylor Series

1. All upper sums are at least as large as all lower sums. That is, for any partitions P_1, P_2 and function $f : [a, b] \to \mathbb{R}$,
$$U_a^b(f, P_1) \geq L_a^b(f, P_2)$$
2. Darboux criterion: The function f is Riemann integrable on $[a, b]$ if and only if for all ε there is a partition P for which $U_a^b(f, P) - L_a^b(f, P) < \varepsilon$.
3. Continuous functions are Riemann integrable (on closed bounded domains).
4. The function f is Riemann integrable on $[a, b]$ with value s if and only if for all ε there is a δ such that $U_a^b(f, P) - s < \varepsilon$ and $s - L_a^b(f, P) < \varepsilon$ whenever $\|P_n\| < \delta$.
5. The Riemann integral has several inadequacies.
6. The nth order Taylor series of $f(x)$ about $x = a$ is given by
$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n$$
7. The nth Taylor remainder term is
$$R_n(x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n \right).$$
8. If $f \in C^{n+1}$ in a neighborhood of a, then $R_n(x) \leq O(|x-a|^{n+1})$ as $x \to a$. More precisely,
$$R_n(x) \leq \max |f^{(n+1)}| \cdot \frac{|x-a|^n}{n!}.$$
The max is taken over the neighborhood and the inequality holds for all points in the neighborhood.

Exercises: Riemann integral DNE or ∞ or $-\infty$ or finite on $[0,1]$

a) $f(x) = \begin{cases} 1 & \text{if } x \neq 1/2 \\ \infty & \text{if } x = 1/2 \end{cases}$

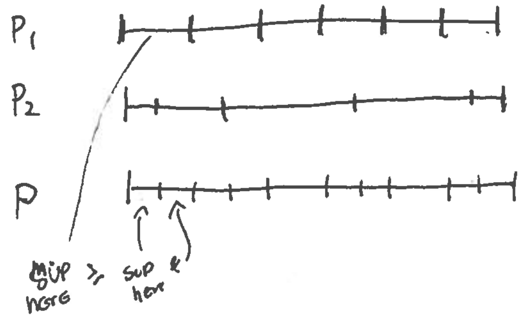
b) $f(x) = \begin{cases} 1 & \text{if } x \leq 1/2 \\ \infty & \text{if } x > 1/2 \end{cases}$

c) $f(x) = \begin{cases} -\infty & \text{if } x \leq 1/3 \\ \infty & \text{if } x \geq 2/3 \\ 0 & \text{owise} \end{cases}$

4) $\forall$ partitions P_1 & P_2, $\forall f: [a,b] \to \mathbb{R}$

$$U_a^b(f, P_1) \geq L_a^b(f, P_2)$$

Proof (gist)



combine partitions

So $U_a^b(f, P_1) \geq U_a^b(f, P) \geq L_a^b(f, P) \geq L_a^b(f, P_2)$.

5) Proof

$\Rightarrow$: If f Riemann integrable $\exists$ P_1 st $U_a^b(f,P_1) - S < \varepsilon/2$

$\exists$ P_2 st $S - L_a^b(f,P_2) < \varepsilon/2$

Consider combinatn of P_1 & P_2. Call it P.

$U_a^b(f,P) - S < \varepsilon/2$ & $S - L_a^b(f,P) < \varepsilon/2$.

So $U_a^b(f,P) - L_a^b(f,P) < \varepsilon$

$\Leftarrow$: Suppose f not Riemann integrable.

$\cancel{\sup_P L_a^b(f,P) \leq \inf_P U_a^b(f,P)}$

$\inf_P U_a^b(f,P) - \sup_P L_a^b(f,P) \geqslant \varepsilon > 0$ for some ε.

So $\forall$ P_1, P_2 $U_a^b(f,P_1) - L_a^b(f,P_2) \geqslant \varepsilon$

Hence $\nexists$ P st $U_a^b(f,P) - L_a^b(f,P) < \varepsilon$ ∎

6) Let $f \in C[a,b]$. f is Riemann integrable

Proof: By Darboux, suffices to show $\forall \varepsilon \exists P$ st $U(f,P)-L(f,P) < \varepsilon$

Fix ε.

As $f \in C[a,b]$, f is uniformly continuous.

Hence $\exists \delta$ st $|x-y|<\delta \Rightarrow |f(x)-f(y)| \leq \frac{\varepsilon}{(b-a)}$

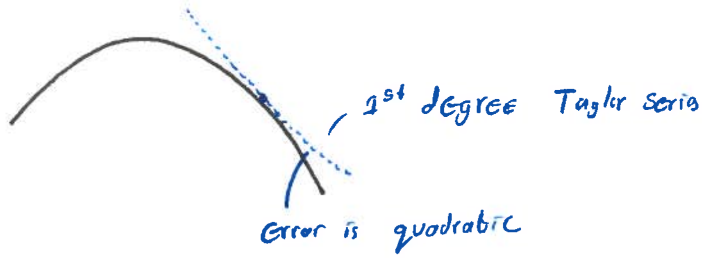
Consider a uniform partition of size $\frac{1}{n}$ where $\frac{1}{n} < \delta$.

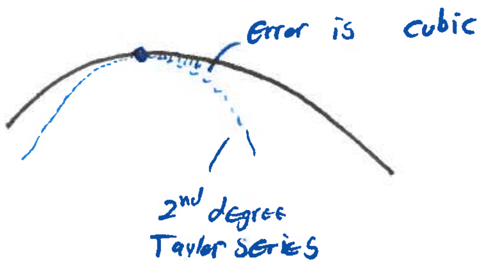
On ~~each~~ i^{th} subinterval $M_i - m_i < \frac{\varepsilon}{(b-a)}$.

$$U(f,P)-L(f,P) = \sum_{i=0}^{n-1} (M_i - m_i)\,\Delta x_i \leq \frac{\varepsilon}{b-a} \sum_{i=0}^{n-1} \Delta x_i = \frac{\varepsilon}{b-a}\, b-a = \varepsilon \quad \blacksquare$$

Taylor Remainder Theorem

An n^{th} order Taylor series has local error on $n+1^{st}$ order





Precise statement:

Let $f \in C^n$ in a nbh of $x=0$.

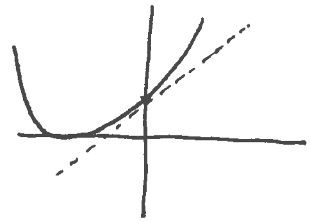
Let $R_n(x) = f(x) - \left[f(0) + \cdots + \frac{f^{(n-1)}}{n-1!}(x-0)^{n-1} \right]$

Then $|R_n(x)| \leq \underbrace{\left(\max_t f^{(n)}(t) \right)}_{\text{max of } n^{th} \text{ deriv sets the constant}} \cdot \frac{|x|^n}{n!}$

$$|R_n(x)| \leq O(|x|^n)$$

Example: Of a function in C^2 such that Error term from 1st order Taylor Series is optimal

$f(X) = (X+1)^2$ at $X=0$



Taylor Series about $x=0$

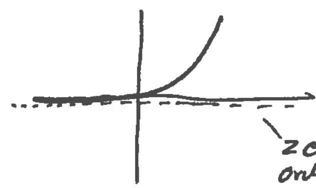
$f(x) \approx 1 + 2X$

Theorem guarantees $|f(x) - (1+2x)| \leq 2 \frac{|x|^2}{2} = |x|^2$ for small x

Actual Error: $(x+1)^2 - (1+2x) = x^2+2x+1 - 1 - 2x = x^2$ ✓

Example: Search for a $f \in C^1$, $f \notin C^2$ such that Error term of 0^{th} order Taylor Series is optimal.

Consider $f(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x \geq 0 \end{cases}$



Taylor series about $x=0$

$f(x) \approx 0 + 0x$

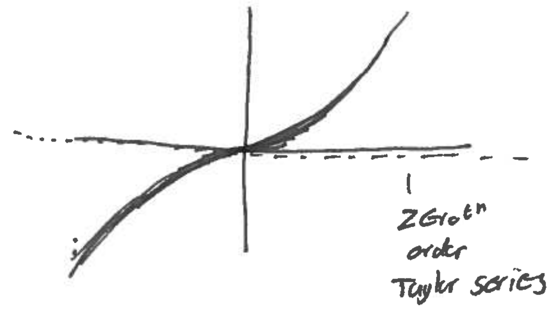
Thm guarantees $|f(x) - 0| \leq 2|x|$

Actual error $|f(x) - 0| \leq |x|^2$ which is better than theorem guarantees.

Example: Want $f \in C^1$, $\notin C^2$ s.t. zeroth order Taylor series error is optimal

Let $f_n(x) = x^{1+\frac{1}{n}}$ for n odd, positive integer

Note $f_n \in C^1$
$f_n \notin C^2$



~~Theorem guarantees~~

~~First order~~
Zeroth order Taylor series

$f_n(x) \approx f_n(0) = 0$

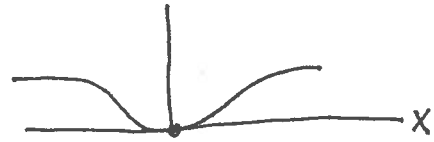
Theorem guarantees: $|f_n(x) - 0| \leq |x|$ on $[-1, 1]$

Actual error: $|f_n(x) - 0| = |x|^{1+\frac{1}{n}}$ which is a tiny bit better

As $n \to \infty$, we reach optimal error estimate
(note $n \to \infty$ limit is $f_\infty(x) = x$)

Example: If $f \in C^\infty$, is infinite Taylor series exact?
or is it merely more accurate than any power of x.

$$f(x) = \begin{cases} e^{-\frac{1}{|x|}} & x \neq 0 \\ 0 & x = 0 \end{cases} \in C^\infty(\mathbb{R})$$



Taylor series about $x=0$ of any order is $f(x) \approx 0$.

Theorem guarantees $|f(x) - 0| \leq C_n |x|^n \quad \forall n$.

Actual error is bdd by $e^{-\frac{1}{|x|}}$ which is faster decaying than any power of x as $x \to 0$

Example of a function nowhere equaling its Taylor series (except at single point)

Application: Order of accuracy of discretization of a derivative.

If f smooth, $\dfrac{f(\Delta x)-f(0)}{\Delta x} \approx f'(0) + O(\Delta x)$. first order

$\dfrac{f(\Delta x)-f(-\Delta x)}{2\Delta x} \approx f'(0) + O(\Delta x^2)$ Second order

First order:

Proof: $f(\Delta x) = f(0) + f'(0)\Delta x + R_1(\Delta x)$

so $\dfrac{f(\Delta x)-f(0)}{\Delta x} = f'(0) + \dfrac{R_1(x)}{\Delta x}$.

We know $|R_1(x)| \leq \max f'' \cdot \dfrac{\Delta x^2}{2}$

So $\dfrac{f(\Delta x)-f(0)}{\Delta x} = f'(0) + O(\Delta x)$

Second order:

$f(\Delta x) = f(0) + f'(0)\Delta x + f''(0)\dfrac{\Delta x^2}{2} + R_2(\Delta x)$

$f(-\Delta x) = f(0) + f'(0)(-\Delta x) + f''(0)\dfrac{\Delta x^2}{2} + \tilde{R}_2(\Delta x)$

$$\frac{f(\Delta x)-f(-\Delta x)}{2\Delta x} = f'(0) + \underbrace{\frac{R_2(\Delta x)-R_2(-\Delta x)}{2\Delta x}}_{\leq \frac{\Delta x^3+\Delta x^3}{\Delta x} = \Delta x^2}$$

So discretization accurate to 2^{nd} order.

Proof:

Taylor Series Remainder

$$f(x) = f(0) + \int_0^x f'(t)\,dt.$$

$\downarrow$ IBP

$$f(x) = f(0) + f'(0)x + \int_0^x (x-t)\, f''(t)\,dt$$

$\downarrow$ IBP

$$f(x) = f(0) + f'(0)x + \dots + f^{(n-1)}(0)\frac{x^{n-1}}{n-1!} + \underbrace{\int_0^x \frac{(x-t)^{n-1}}{n-1!} f^{(n)}\,dt}_{\text{Remainder term.}}$$

~~Use intermediate Value Theorem~~

Bound

$$\leq \max f^{(n)} \int_0^x \frac{(x-t)^{n-1}}{n-1!}\,dt$$

$$= \max f^{(n)} \left(-\frac{(x-t)^n}{n!} \Big|_0^x \right)$$

$$= \max f^{(n)} \frac{x^n}{n!}.$$