11 September 2014
Analysis I
Paul E. Hand
hand@rice.edu

Day 6 — Summary — Convexity, Inverse Function Theorem, Riemann Integration

1. A function is convex if for all $t \in (0, 1)$ and for all points a and b,

$$f\Big((1-t)a + tb\Big) \leq (1-t)f(a) + tf(b).$$

   It is strictly convex if this inequality is strict.

2. If $f''(x) > 0$ in an interval, then f is strictly convex in the interval.

3. A continuous, strictly increasing function has an inverse that is continuous and strictly increasing.

4. A differentiable, strictly increasing function has an inverse that is differentiable and strictly increasing. The derivative of the inverse is the inverse of the derivative:

$$\frac{dy}{dx}(x) = \left(\frac{dx}{dy}(y)\right)^{-1}$$

Exercise:

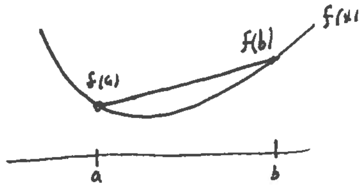
Find a function $f \in C^\infty(\mathbb{R})$ such that $f \equiv 0$ outside of $[-1, 1]$.

Draw it first.

Put it in form $f(x) = e^{g(x)}$

Try to squish e^{-x^2} into something finite.

1) f is convex if it is below its secant line segments



$$f\big(\underbrace{(1-t)a+tb}_{\text{convex combination of } a \,\&\, b}\big) \le \underbrace{(1-t)f(a)+t f(b)}_{\text{convex combination of } f(a) \,\&\, f(b)}$$

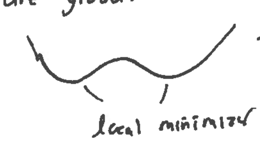
Strictly convex if strict inequality

Examples:

Convex on $\mathbb{R}$	$f(x) =$ ~~[illegible]~~ $ax+b$
Not strictly convex on $\mathbb{R}$	$f(x) = \lvert x \rvert$
Strictly convex	$f(x) = x^2$

Applications: - Convex optimization $\min f(x)$ is "Easy" if f is convex. Even if f isnt smooth

- Note: all local minimizers are global

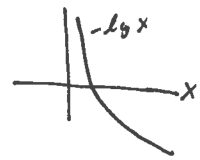
Cant have:  - not convex

-

Application: Arithmetic-geometric mean

$$\frac{a+b}{2} \geq \sqrt{ab} \quad \text{for } a,b \geq 0$$

Why? $-\log(x)$ is convex



$$-\log\left(\frac{a+b}{2}\right) \leq \frac{-\log a - \log b}{2} \quad \text{by convexity.}$$

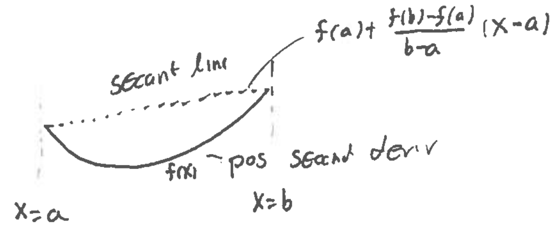
$$\log\left(\frac{a+b}{2}\right) \geq \frac{\log a + \log b}{2}$$

$$\frac{a+b}{2} \geq a^{1/2} b^{1/2} \quad \blacksquare$$

Application: Jensens inequality in probability.

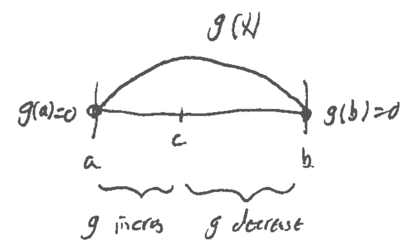
2) Theorem: If $f'' > 0$ on $[a,b]$, then f convex on $[a,b]$

Visually:



Proof: Consider difference of secant line & f

$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) - f(x)$$



Show $g' > 0$ for $x \in (a,c)$
$g' < 0$ for $x \in (c,b)$.

Compute, $g'(x) = \frac{f(b)-f(a)}{b-a} - f'(x)$

$$g'(x) = f'(c) - f'(x)$$

$$= f''(d)(c-x)$$

> 0 for $x > c$
< 0 for $x < c$

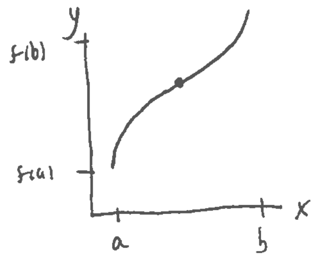
2 a)

Example of function that is strictly convex, yet f'' isnt always positive.

$f(x) = x^4$.

3) Theorem: If $f \in C[a,b]$ is strictly increasing, then f^{-1} exists and is continuous $C[f(a), f(b)]$ and strictly increasing

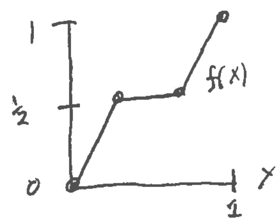
Visual:



Meaning of f^{-1}: $f^{-1}(y) = x$ such that $f(x)=y$.

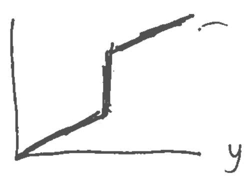
For f^{-1} to be well defined, there has to be exactly one x such that $f(x)=y$ $\forall\, y \in [f(a), f(b)]$

Nonexample:



$f(x)$ is increasing but not strictly.

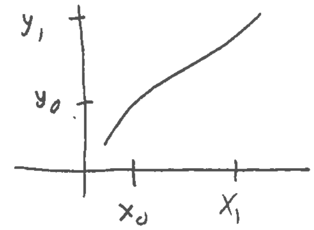
$f^{-1}(y)$ is not well-defined because $\exists$ many x such that $f(x) = \frac{1}{2}$.



$f^{-1}(y)$ is not a function

Proof:

Let $y_0 = y(x_0)$
$y_1 = y(x_1)$



Note: $$\frac{x(y_1) - x(y_0)}{y_1 - y_0} = \frac{x_1 - x_0}{y(x_1) - y(y_0)} = \frac{1}{\frac{y(x_1) - y(x_0)}{x_1 - x_0}}$$

Because $x(y)$ & $y(x)$ is continus,

$$x'(y_0) = \lim_{y_1 \to y_0} \frac{x(y_1) - x(y_0)}{y_1 - y_0} = \lim_{x_1 \to x_0} \frac{x_1 - x_0}{y(x_1) - y(x_0)} = \frac{1}{y'(x_0)}$$

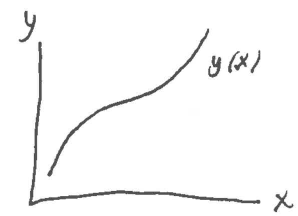
4) Inverse function Theorem

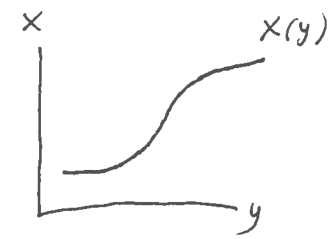
Idea: A diffable function that has ~~strictly increasing~~ positive derivative over a region has a diffable inverse ~~that has~~ and deriv of inverse is inverse of derivatives

Precise: If $y(x)$ diffable on $[a,b]$, $y'(x) > 0 \ \forall \ x \in [a,b]$

then $\exists$ $x(y)$ diffable on $[y(a), y(b)]$ with $\frac{dx}{dy}(y) = \frac{1}{\frac{dy}{dx}(y)} = \frac{1}{\frac{dy}{dx}(x(y))}$

Visual:





Why doesnt theorem assume that function is merely increasing?

Such a function has a continuous inverse but maybe not a diffable one.

$y(x) = x^3$ has inverse $x(y) = y^{1/3}$ not diffable at $y=0$