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Analysis I
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Day 22 — Summary — Definition of Riemann integral by limits of step functions

1. A step function from $[a, b] \to E$, where E is a normed vector space, is a function of the form

$$f(x) = w_i \text{ for } a_{i-1} < t < a_i,$$

   where $a = a_0 \leq a_1 \leq \ldots \leq a_n = b$ is a partition of $[a, b]$. Denote the set of step functions as $\mathrm{St}([a, b], E)$.

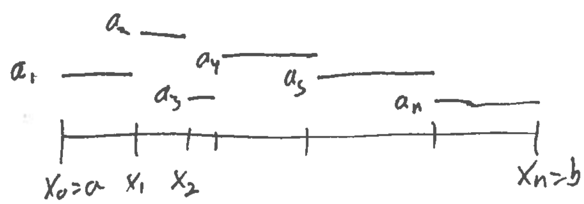
2. The integral of a step function on $[a, b]$ is defined as $I(f) = \sum_{i=1}^{n}(a_i - a_{i-1})w_i$.

3. $\mathrm{St}([a, b], E)$ is a subspace of the space of all bounded maps from $[a, b]$ into E. The operator I is a linear operator from this subspace to E with bound $b - a$. That is, $\|I(f)\|_E \leq (b - a)\|f\|$.

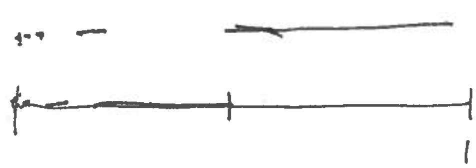
4. The integral operator I can be extended to the closure of $\mathrm{St}([a, b], E)$. We will call this closure the space of regulated maps, $\mathrm{Reg}([a, b], E)$.

5. The closure of $\mathrm{St}([a, b], E)$ contains $C^0([a, b], E)$. It also contains the class of piecewise continuous functions.

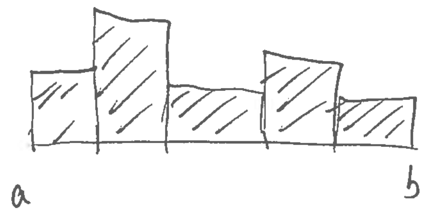
1) Step function:



Not a step function $\begin{cases} 1 & \frac{1}{n+1} < x < \frac{1}{n} \text{ for } n \text{ odd} \\ 0 & \frac{1}{n+1} < x < \frac{1}{n} \text{ for } n \text{ even} \end{cases}$



2)



$$I(f) = \sum_{i=1}^{n} (a_i - a_{i-1}) w_i$$