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Analysis I
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Day 18 — Summary — Series

1. If $\{x_n\}$ is a sequence in a normed vector space, we define the infinite sum $\sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n$. The infinite series converges if this sum exists. We say that an infinite series diverges if the partial sums are unbounded.
2. Comparison test. Let $\sum a_n$ and $\sum b_n$ be series of real numbers. If $\sum b_n$ converges and $0 \leq a_n \leq b_n$ for sufficiently large n, then $\sum a_n$ converges.
3. Ratio test. Let $\sum a_n$ be a series of nonnegative real numbers, and let $0 < c < 1$ be such that $a_{n+1} \leq c a_n$ for sufficiently large n. Then $\sum a_n$ converges.
4. Integral test. Let f be a decreasing function over all real numbers ≥ 1. The infinite series $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_a^\infty f(x)dx$ exists and is finite. Note that $\int_a^\infty f(x)dx$ is defined as $\lim_{M\to\infty} \int_1^M f(x)dx$.
5. Let $\sum a_n$ be a series of numbers. If $\sum |a_n|$ converges, then $\sum a_n$ converges. The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.
6. Let $\{a_n\}$ be a sequence of numbers monotonically decreasing to zero. The alternating series $\sum (-1)^n a_n$ converges .

Warmup:

Provide a Cont function f over ℓ^∞ and a closed S. such that $\max f$ is finite & not achieved

1)

Converges:

$\sum_{n=1}^{\infty} a^n$ for $|a|<1$ (direct computation)

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p>1$ (~~integral~~ ~~ratio~~ integral test)

Diverges:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p \leq 1$

$\sum_{n=1}^{\infty} a^n$ for $\cancel{|a|>1}$ $a \in (-\infty, 1) \cup [1, \infty)$

Exampl: let $S_N = \sum_{i=0}^{N} a^n$

$$S_N = 1 + a + \cdots + a^N$$
$$aS_N = a + \cdots + a^N + a^{N+1}$$
$$(a-1)S_N = a^{N+1} - 1$$
$$S_N = \frac{1-a^{N+1}}{1-a}$$
$$\lim_{N\to\infty} S_N = \frac{1}{1-a} \quad \text{if } |a|<1$$

2) Let ~~$a_n, b_n \geq$~~ $0 \leq a_n \leq b_n$ for $n \geq N$.

Then $\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges

$\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum b_n$ diverges

Example:

$$\sum_{n=1}^{\infty} \frac{\log n}{n} \text{ diverges by comparison to } \sum_{n=1}^{\infty} \frac{1}{n}.$$

Example: $\sum_{n=1}^{\infty} \frac{\log^3 n}{n^2}$ converges, by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$

Becaus $\lim_{n \to \infty} \frac{\log^3 n}{\sqrt{n}} = 0$, $\exists N$ st $\log^3 n \leq \sqrt{N}$

Example: $\sum_{n=1}^{\infty} n^3 e^{-n}$ converges by comp to $\sum_{n=1}^{\infty} e^{-n/2}$

3) If $0 \le a_{n+1} \le c a_n$ for $c \in (0,1)$ and $n \geq N$,

then $\sum_{n=1}^{\infty} a_n$ converges.

pf: Compare to $\sum_{n=1}^{\infty} c^n$ (or $a_N \sum_{n=1}^{\infty} c^{(n-N)}$)

Example: $\cancel{\sum_{n=1}^{\infty} \frac{1}{n^2} = 1}$

$\cancel{\sum_{n=1}^{\infty} \frac{1}{n^2}}$ ~~Note~~ $\cancel{a_n = \frac{1}{n^2}}$

$\cancel{\frac{a_{n+1}}{a_n} =}$

$\sum_{n=1}^{\infty} \frac{b^n}{n!}$ converges $a_n = \frac{b^n}{n!}$ $\frac{a_{n+1}}{a_n} = \frac{b^{n+1}}{n+1!} \frac{n!}{b^n} = \frac{b}{n+1} \to 0$

Ratio test applies for $c = 1/2$, e.g.

5) Let $a_n \in \mathbb{R}$ $\quad \sum_{n=1}^{\infty} |a_n|$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges

Idea: worst case of sum is when everything adds (no cancellation)

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$$

Proof: Let $S_N = \sum_{n=1}^{N} a_n$ $\quad$ Let $T_N = \sum_{n=1}^{N} |a_n|$

$$|S_N| = \left| \sum_{n=1}^{N} a_n \right| \leq \sum_{n=1}^{N} |a_n| = T_N \text{ bounded.}$$

So S_N is bdd and $\uparrow$. So S_N conv.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ conv bc $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv