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Analysis I
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Day 11 — Summary — Equivalent Norms and Banach Spaces

1. Definition: Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent on a vector space V if there exists $c, C > 0$ such that
$$c\|x\|_b \leq \|x\|_a \leq C\|x\|_b \; \forall x \in V.$$

2. All norms on finite dimensional vectors spaces, e.g. $\mathbb{R}^n$, are equivalent.

3. In infinite dimensional vector spaces, some pairs of norms are not equivalent.

4. Definition: A sequence x_n in a normed vector space is Cauchy if
$$\forall \varepsilon \; \exists N \text{ such that } n, m \geq N \Rightarrow \|x_n - x_m\| < \varepsilon.$$

5. In a normed vector space, we say that x_n converges to x if $\forall \varepsilon \; \exists N$ such that $n \geq N \Rightarrow \|x_n - x\| < \varepsilon$. We write this as $\lim_{n\to\infty} x_n = x$

6. Definition: A vector space is complete if any Cauchy sequence converges to an element in the set.

7. Definition: A Banach space is a complete normed vector space.

8. Definition: $\mathbb{R}^n$ is a Banach space under the ℓ_∞ norm. By equivalence of norms on finite dimensional spaces, it is a Banach space under any norm.

1) Equivalent norms

Background: Norms are used to define a notion of convergence, completeness, open sets, etc.

Some pairs of norms will produce same notion of convergence, completeness. Some will produce different notions.

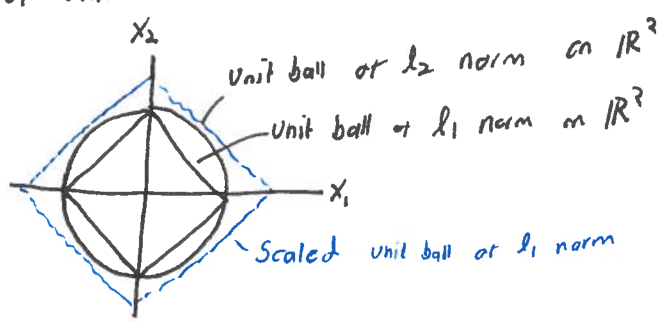
Eg a seq may converge in one norm but not another

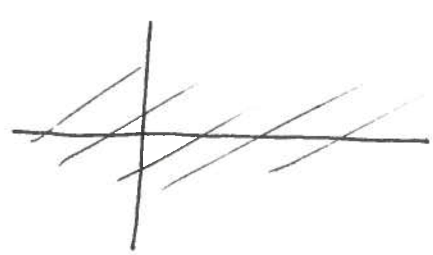
Idea: Two norms are equivalent when all norm-defined notions are the same.

Eg a seq x_n conv in one norm $\Rightarrow$ seq x_n converges in other norm

Dfn: $\|\cdot\|_a$ Equivalent to $\|\cdot\|_b$ if $\exists\ c, C$ st $c\|x\|_b \leq \|x\|_a \leq C\|x\|_b$

Visualization: Unit ball of each norm is contained in some multiple of unit ball of the other





3) Example: Consider vector space ℓ_1

($\text{sequences } X_i \in \mathbb{R}$ st $\sum_{i=1}^{\infty} |X_i| < \infty$).

The ℓ_1 norm and ℓ_∞ norm are not equivalent.

Pf: ~~Suffices to exhibit a sequence of~~

Suffices to exhibit for any N an X such that $\|X\|_\infty = 1$ $\|X\|_1 = N$

Fix N. Let $X_i = \begin{cases} 1 & i \leq N \\ 0 & i > N \end{cases}$. $\|X\|_\infty = 1$ $\|X\|_1 = N$ ∎

Exercise: ~~Find a seq $\|X\|_1 = 1$ $\|X\|$~~

Exhibit a collection of X showing that ℓ_1 and ℓ_2 norm are not equivalent.

Example: In $\mathbb{R}^n$, $\|\cdot\|_1$ equivalent to $\|\cdot\|_\infty$

Proof: ~~Need to show~~ $\cancel{c\|X\|_\infty \leq \|X\|_1 \leq C\|X\|_\infty}$

We will show $\|X\|_\infty \leq \|X\|_1 \leq n\,\|X\|_\infty$

Left inequality: immediate

Right inequality: $\|X\|_1 = \sum_{i=1}^{n} |X_i| \leq \sum_{i=1}^{n} \max |X_i| = n \max |X_i|$

$= n\,\|X\|_\infty .$

Exercise: Find C such that $\|x\|_1 \leqslant C\|x\|_2$ for $x \in \mathbb{R}^n$

— C_1 such that $\|x\|_2 \leqslant C_1\|x\|_1$

Warmup:

b) Fix an N.
Find a sequence X such that $||X||_2 = 1$ and $||X||_1 = N$

a) Fix an N
Find a seq X s.t. $||X||_\infty = 1$ and $||X||_1 = N$

8) $(\mathbb{R}^n, \|\cdot\|_\infty)$ is complete.

Proof:

Consider X^i a Cauchy seq in $\mathbb{R}^n$ under $\|\cdot\|_\infty$

$\forall \varepsilon \; \exists N$ st $n,m \geq N \implies \|X^n - X^m\|_\infty < \varepsilon$

As $|X_j^n - x_j^m| \leq \|X^n - x^m\|_\infty$ we hav

$\forall \varepsilon \; \exists N$ st $n,m \geq N \implies \|X_j^n - x_j^m|_\infty < \infty.$

So each component is Cauchy and has limit $X_j^\infty \in \mathbb{R}$.

~~Then~~

Remains to show $X^i \to X^\infty$ under $\|\cdot\|_\infty$.

$\forall \varepsilon \; \exists N_j$ st $|X_j^n - x_j^\infty| < \varepsilon \;\; \forall \; n \geq N$

Fix ε. Let $N = \max N_1 \ldots N_n$. Then $\|X^n - x^\infty\| < \varepsilon$

7) Example: Space that isn't complete wrt a norm

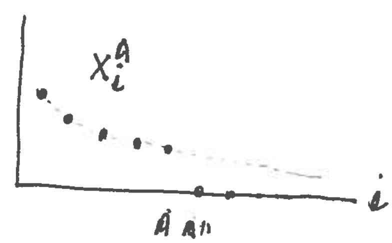
ℓ_1 not complete wrt ℓ_∞ norm

$$V = \ell_1 = \{ X \text{ sequence} \mid \sum_{i=1}^{\infty} |X_i| < \infty \}$$

Pf: Suffices to exhibit ~~Cauch~~ seq $X^i \in V$ that is Cauchy wrt ℓ_∞ but does not converge to anything in V.

$$\text{Let } X^n_i = \begin{cases} \frac{1}{i} & \text{if } i \leq n \\ 0 & \text{o'wise} \end{cases}$$

$$= \mathbb{1}_{[1,n]}(i) \cdot \frac{1}{i}$$



Note: $\|X^n - X^m\|_\infty = \frac{1}{n \wedge m}$

This seq is Cauchy. Fix ε. Let $N > \frac{1}{\varepsilon}$.

$\forall\, n, m \geq N \quad \|X^n - X^m\|_\infty \leq \frac{1}{N} < \varepsilon.$

There is no $X^\infty \in \ell_1$ such that $X^n \to X^\infty$ in ℓ_∞.

Informally: B/c $X^n \to \{\frac{1}{i}\}_{i=1}^{\infty}$ which is not in ℓ_1

Formally: Each $X^n \in \ell_\infty$. ℓ_∞ is complete wrt ℓ_∞ norm. $X^n \to \{\frac{1}{i}\}_{i=1}^{\infty}$ in ℓ_∞ norm.

Limits are unique, so if $X^n \to X^\infty \in \ell_1 \subset \ell_\infty$, $X^\infty = \{\frac{1}{i}\}_{i=1}^{\infty} \notin \ell_1$. ∎

Exercise: ℓ_∞ is complete under the ℓ_∞ norm.