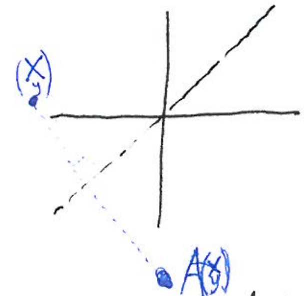
1) Practice for Quiz 2 18.085

Let A be the 2×2 matrix such that $A\binom{x}{y}$ is the reflection of $\binom{x}{y}$ through $y=x$



without finding A, find two independent eigenvectors & their eigenvalues.

Solution

We seek points $\binom{x}{y}$ such that $A\binom{x}{y}=\lambda\binom{x}{y}$

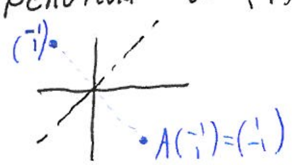
Notice that any point along $y=x$ is its own reflection

e.g. $A\binom{1}{1}=\binom{1}{1}$.

Thus $\binom{1}{1}$ is eigenvector w/ eigenvalue 1

We seek another eigenvector. Because reflection flips the part of $\binom{x}{y}$ perpendicular to $\binom{1}{1}$, we try $\binom{x}{y}=\binom{-1}{1}$

$$A\binom{-1}{1}=\binom{1}{-1}$$



Thus

$\binom{-1}{1}$ is eigenvector w/ eigenvalue -1.

2)

A is a symmetric matrix with eigenvalues $-3, 1, 2,$ corresponding to eigenvectors $\begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}1\\-1\\0\end{pmatrix} \begin{pmatrix}1\\1\\-2\end{pmatrix}$, respectively. Write down an eigenvalue decomposition of A.

<u>Solution</u>

As A is symmetric, it has orthonormal basis of eigenvectors. v_1, v_2, v_3.

$$A = V \Lambda V^t$$

where

$$V = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Observing that the given eigenvectors are orthogonal to each other, we find v_1, v_2, v_3 by normalization

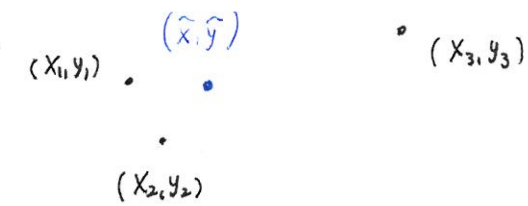
$$V = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Thus

$$A = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}$$

3) Set up a least squares problem to find the 2d point $(\hat{x}, \hat{y})$ that minimizes the sum of the square distances to (x_1, y_1), (x_2, y_2), (x_3, y_3). Solve it by hand.



Solution

The square distance to (x_i, y_i) from $(\hat{x}, \hat{y})$ is

$$(x_i - \hat{x})^2 + (y_i - \hat{y})^2$$

The sum of square distances is

$$(\hat{x} - x_1)^2 + (\hat{y} - y_1)^2 + (\hat{x} - x_2)^2 + (\hat{y} - y_2)^2 + (\hat{x} - x_3)^2 + (\hat{y} - y_3)^2$$

We write this expression as $\|Au - b\|^2$ where $u = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$ is a vector of unknown coefs.

Each row of $Au - b$ is squared, so we see A is 6×2

Identify
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} \qquad u = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

Least squares problem:

$$\min_u \|Au - b\|^2 \quad \text{with}$$

Solution to least squares problem given by normal equations

$$A^t A \hat{u} = A^t b.$$

Computing,

$$A^t A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A^t b = \begin{pmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \end{pmatrix}$$

So $$\hat{u} = \begin{pmatrix} \frac{x_1 + x_2 + x_3}{3} \\ \frac{y_1 + y_2 + y_3}{3} \end{pmatrix}.$$

4)

Suppose A is the 3×3 matrix such that Ax is a rotation of $\pi/6$ radians about the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$. Find the condition number of A without finding A itself.

Solution

The condition number of A is $\dfrac{\sigma_{max}(A)}{\sigma_{min}(A)}$.

Recall the geometric interpretation of the i^{th} singular value, σ_i, as the length stretching factor of Av_i, where v_i is the i^{th} right singular vector and u_i is i^{th} left-singular vector

$$A v_i = \sigma_i u_i$$

Thus $\sigma_i = \| A v_i \|$ as $\|u_i\| = 1$.

Because rotation preserves length,
$\|Ax\| = \|x\|$ for all x.

Hence all singular values are 1.

Thus condition number is 1.

5) What is the range, rank, and null space of A, given by the following SVD

$$A = \begin{pmatrix} 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} \\ \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix}$$

Solution

The rank is the number of nonzero singular values.

The singular values are 2 & 0.

$\text{Rank}(A) = 1$

The range of A is the span of the left singular vectors corresponding to nonzero singular values

The left singular vector for $\sigma_1 = 2$ is $U_1 = \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{pmatrix}$.

The range is $\left\{ \lambda \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{pmatrix} \text{ for all } \lambda \right\}$

The null space of A is the span of the right singular vectors corresponding to zero singular values

The right sing. vector for $\sigma_2 = 0$ is $V_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

The null space is $\left\{ \lambda \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \text{ for all } \lambda \right\}$

6) Write out the matrix F_4 by hand.
Use it to compute $fft\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$.

Solution

$$(F_4)_{jk} = e^{2\pi i (j-1)(k-1)/4} \qquad \begin{array}{l} j=1\cdots 4 \\ k=1\cdots 4 \end{array}$$

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{2\pi i/4} & e^{2\pi i 2/4} & e^{2\pi i \cdot \frac{3}{4}} \\ 1 & e^{2\pi i\, 2/4} & e^{2\pi i\, 4/4} & e^{2\pi i\, 6/4} \\ 1 & e^{2\pi i\, 3/4} & e^{2\pi i\, 6/4} & e^{2\pi i\, 9/4} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

If $x = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$ $\qquad \hat{x} = F_4^* x$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}2\\0\\-2\\0\end{pmatrix}$$