Name:

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18.085
Paul E. Hand
hand@math.mit.edu

Quiz 3 Practice

1. (20 points) Consider $\frac{dy}{dt} = -y^2$. Let y_n be the computed value of $y(n\Delta t)$. Write out an explicit formula for y_{n+1} in terms of y_n using
   (a) Forward Euler
   (b) Backward Euler
   (c) Trapezoidal rule

2. Find values of c_0, c_1, c_2 such that $c_0 u(0) + c_1 u(\Delta x) + c_2 u(2\Delta x)$ is a first order approximation of $u''(0)$.

3. Consider the 2π periodic boundary value problem:

$$-\frac{d}{dx}\left(\rho(x)\frac{du}{dx}\right) = f(x)$$
$$u(0) = u(2\pi)$$
$$u'(0) = u'(2\pi)$$

   Assume $\rho(0) = \rho(2\pi)$. Show that any solution to this equation satisfies the following weak form:

$$\int_0^{2\pi} \rho(x)\frac{du}{dx}\frac{d\phi}{dx}dx = \int_0^{2\pi} f(x)\phi(x)dx \text{ for all } 2\pi \text{ periodic } \phi(x)$$

4. Consider the boundary value problem with Dirichlet boundary conditions:

$$-\frac{d}{dx}\left(\rho(x)\frac{du}{dx}\right) = 1$$
$$u(0) = 0$$
$$u(1) = 0$$

   where $\rho(x) = \begin{cases} 1 & \text{if } x < 1/2 \\ 2 & \text{if } x > 1/2 \end{cases}$

   (a) Write out the weak form of this boundary value problem
   (b) Let $N = 3$, $\Delta x = \frac{1}{4}$, $x_i = i\Delta x$. For $i = 1, 2, 3$, let $\phi_i(x)$ be the piecewise linear functions that satisfy $\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Write out the linear system that needs to be solved under a finite element method with basis functions ϕ_1, ϕ_2, ϕ_3

5. What is the complex Fourier series expansion of $f(x) = 1 + \sin(4x)$?

6. What is the solution to

$$\begin{aligned} -\frac{d^2u}{dx^2} &= e^{4ix} + e^{-3ix} \\ u(0) &= u(2\pi) \\ u'(0) &= u'(2\pi) \end{aligned}$$

7. Solve

$$\begin{aligned} -\frac{d^2u}{dx^2} &= \delta\left(x - \frac{L}{3}\right) \\ u(0) &= 0 \\ u(L) &= 0 \end{aligned}$$

   (a) directly

   (b) by a Fourier Sine series

8. The functions $\phi_n(x) = \sin\frac{n\pi x}{L}$ for $n = 1, 2, 3, \cdots$ form an orthogonal basis of functions from $x = 0$ to $x = L$. This means $f(x) = 1$ can be written as a sum of sines. Find the expansion of $f(x) = 1$ in this basis.