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18.085
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Quiz 2

Rules: 90 minutes. Open notes, open book, closed electronics.
Please show all of your work. There are 5 problems.

1. (20 points) Let A be the 3×3 symmetric matrix such that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the nearest point to $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ on the plane $x + y = 0$. Is A positive definite? Is A positive semi-definite? Justify your answers.

Because A is symmetric, there are 3 orthonormal eigenvectors.

Observe that any point on the plane $x+y=0$ is an eigenvector of eigenvalue 1.

Two independent eigenvectors are $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

The third eigenvector must be perpendicular to both these: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

This eigenvector has eigenvalue 0 because $A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

A is not pos. def because there is a 0 eigenvalue.

A is pos. semidef because all eigenvalues are nonnegative.

2. (20 points) Let $A = \tilde{U}\tilde{\Sigma}\tilde{V}^*$ be a reduced SVD of A. The pseudoinverse of A is $A^+ = \tilde{V}(\tilde{\Sigma}^{-1})\tilde{U}^*$.

(a) (10 points) Find the pseudoinverse of

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

(b) (10 points) Use the normal equations to show that the minimizer of $\min_u ||Au - b||^2$ is given by $\hat{u} = A^+ b$.

Hint: You may need the fact that $(BCD)^{-1} = D^{-1}C^{-1}B^{-1}$ for invertible matrices B, C, and D.

a) First we find the reduced SVD of $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$.

Note that the columns are orthogonal, so A is almost $\tilde{U}$ in a reduced SVD. We normalize the columns of A to get $\hat{U}$ and $\hat{\Sigma}$

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}}_{\hat{U}} \underbrace{\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}}_{\hat{\Sigma}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\hat{V}^*}$$

Hence $A^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$

b) The minimizer $\hat{u}$ satisfies $A^*A\hat{u} = A^*b$.

We show $\hat{u} = A^+b$ satisfies this eqn:

Want to show $A^*AA^+b = A^*b$.

Suffices to show $A^*AA^+ = A^*$

Note $(\hat{U}\tilde{\Sigma}\hat{V}^*)^* = \hat{V}\hat{\Sigma}^*\hat{U}^* = \hat{V}\hat{\Sigma}\hat{U}^*$ because $\hat{\Sigma}$ is real, diagonal

$A^*AA^+ = \hat{V}\hat{\Sigma}\cancel{\hat{U}^*\hat{U}}\hat{\Sigma}\cancel{\hat{V}^*\hat{V}}\hat{\Sigma}^{-1}\hat{U}^*$

$= \hat{V}\hat{\Sigma}\cancel{\hat{\Sigma}\hat{\Sigma}^{-1}}\hat{U}^*$

$= \hat{V}\hat{\Sigma}\hat{U}^* = A^*.$

3. (20 points) The fft of the signal x is

$N = 12$

$$\hat{X} = \begin{pmatrix} -12 \\ 0 \\ 0 \\ -6i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6i \\ 0 \\ 0 \end{pmatrix} \qquad \frac{\hat{X}}{N} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -i/2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i/2 \\ 0 \\ 0 \end{pmatrix}$$

Find x and sketch its real and imaginary parts.

Suggestion: Instead of writing out all 12 components of x, present a formula for the jth coefficient of x. Simplify your formula by expanding complex exponentials in terms of sines and cosines.

$$X = F_N\left(\frac{\hat{X}}{N}\right) \qquad F_N = \begin{pmatrix} | & | & & | \\ V_0 & V_1 & \cdots & V_{N-1} \\ | & | & & | \end{pmatrix}$$

$$= -1 \cdot V_0 - \frac{i}{2} \cdot V_3 + \frac{i}{2} V_9$$

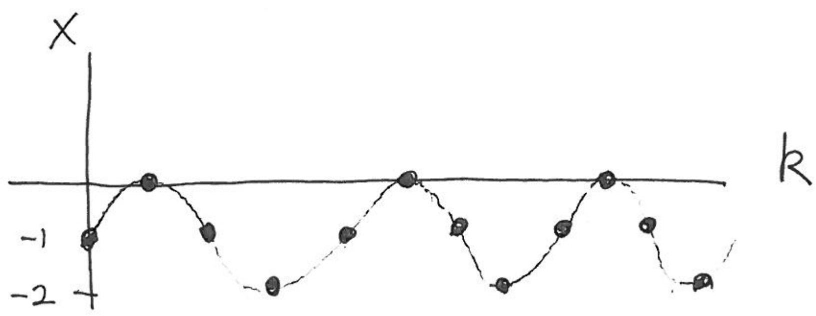
$$V_j(k) = e^{2\pi i j k/N}$$

$$X(k) = -1 - \frac{i}{2} e^{2\pi i 3k/12} + \frac{i}{2} e^{2\pi i 9k/12}$$

$$= -1 - \frac{i}{2} e^{2\pi i 3k/12} + \frac{i}{2} e^{-2\pi i 3k/12} \qquad e^{ix} = \cos x + i \sin x$$

$$= -1 + \sin 2\pi \cdot 3k/12$$

3 oscillating as k varies by 12



4. (20 points)

(a) (10 points) Find the condition number of

$$A = \begin{pmatrix} 1+\varepsilon & 1 \\ 1 & 1+\varepsilon \end{pmatrix}$$

(b) (10 points) The Fourier matrix for $N = 4$ is

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Find the condition number of F_4.
Hint: What is the SVD of F_4? Recall that the columns of F_4 come from the Fourier basis.

a) $\text{Cond}(A) = \dfrac{\sigma_{max}}{\sigma_{min}}$. Symmetric matrix $\Rightarrow$ $\sigma_i = |\lambda_i|$.

Find Eigenvalues of A

$\det(A - \lambda I) = 0 \Rightarrow ((1+\varepsilon)-\lambda)^2 - 1 = 0$

$(1+\varepsilon) - \lambda = \pm 1$

$\lambda = 1 + \varepsilon \pm 1$

$= \begin{cases} 2+\varepsilon \\ \varepsilon \end{cases}$

If ε small, $\lambda_{max} = 2+\varepsilon$

$\lambda_{min} = \varepsilon$

$\text{Cond}(A) = \dfrac{2+\varepsilon}{\varepsilon}$

b) F_4 has orthogonal columns of magnitude 2.

Hence SVD of F_4 is $\left(\frac{1}{2}F_4\right)\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

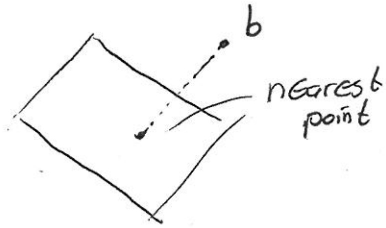
All singular values are 2.

$\text{Cond}(F_4) = \dfrac{\sigma_{max}}{\sigma_{min}} = \dfrac{2}{2} = 1$

5. (20 points)

(a) (10 points) Set up a least squares problem to find the nearest point on the plane $x + y + z = 0$ to the point $(-2, 3, 5)$.

(b) (10 points) Repeat, but for the plane $x + y + z = 1$.



a) Express an arbitrary point on plane using two unknown parameters.

That is, find a matrix A such that the plane $x+y+z=0$ is the range of A.

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The nearest point to $b = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$ on the plane is $A\hat{v}$ where $\hat{v}$ is minimizer of $\min_{v} \|Av - b\|^2$

b) We also express any point on the plane $x+y+z=1$ in terms of two unknowns $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Every point on the plane $x+y+z=1$ is of the form

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$ for some v_1, v_2

The nearest point to b is given by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \hat{v}$

where $\hat{v}$ is minimizer of

$$\min_{v} \left\| Av + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} \right\|$$

$$= \min_{v} \left\| \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} v - \begin{pmatrix} -3 \\ 3 \\ 5 \end{pmatrix} \right\|.$$