18.085 PSET 2 Solns

Paul Hand

1) A is 6×8, so its rank is at most 6.

The 1^{st}, 3^{rd}, 4^{th}, 5^{th}, 7^{th}, 8^{th} columns are independent.

Suppose $$c_1\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}+c_4\begin{pmatrix}0\\0\\-1\\0\\0\\1\end{pmatrix}+c_5\begin{pmatrix}0\\-1\\0\\0\\1\\0\end{pmatrix}+c_6\begin{pmatrix}0\\0\\0\\-1\\-1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

6^{th} row says: $c_4 = 0$

3^{rd} row says: $c_3 - c_4 = 0$, so $c_3 = 0$

4^{th} row says: $-c_6 = 0$

5^{th} row says: $c_5 - c_6 = 0$, so $c_5 = 0$

2^{nd} row says: $c_2 - c_5 = 0$, so $c_2 = 0$

1^{st} row says: $c_1 = 0$.

So $c_1 = c_2 = \cdots = c_6 = 0$. Hence, these vectors are independent and the rank is at least 6.

The rank of A is thus 6.

2) a $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

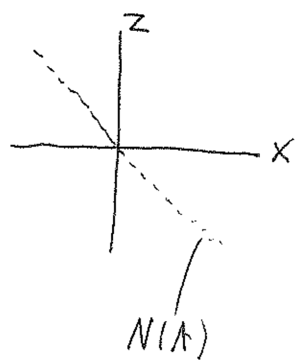
says

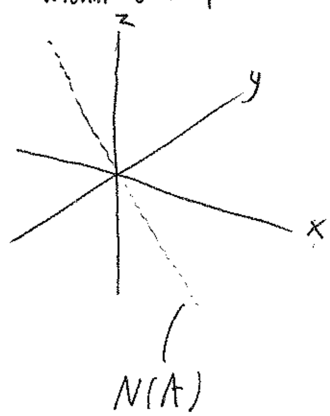
$$x + z = 0$$
$$y = 0$$

The null space is all (x,y,z) satisfying these two equations.

$y = 0$ } xz plane

$x + z = 0 \Rightarrow z = -x$ within that plan





$N(A)$ is a line.

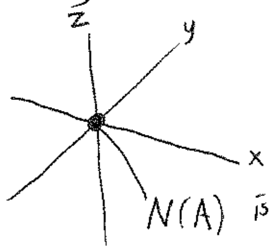
$N(A) = \{ (x, 0, -x), \text{ for all } x \}$

b)

$$\begin{pmatrix}1&0&1\\0&1&0\\1&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\left.\begin{array}{l} x+z=0 \\ \left.\begin{array}{l} y=0 \\ x+y=0 \end{array}\right\} \Rightarrow x=0 \end{array}\right\} z=0.$$

The only point satisfying these eqns is $\begin{pmatrix}0\\0\\0\end{pmatrix}$.



N(A) is a single point

3)

Rewrite the Equation as

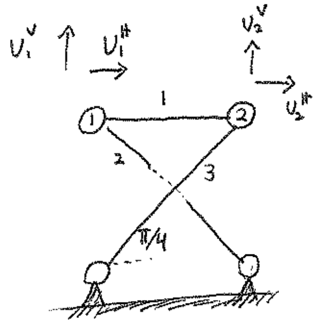
$$\begin{pmatrix} 6 \\ 4 \\ 5 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix}$$

$$b = A \quad X$$

Using matlab, $X = A \backslash b$

$$X = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

4)



LHS is change in bar length given displacements $U_1^H\ U_2^H\ U_1^V\ U_2^V$

RHS says those changes are 0.

Bar 1: $-U_1^H + U_2^H = 0$

Bar 2: $-U_1^H \cos\pi/4 + U_1^V \sin\pi/4 = 0$

Bar 3: $U_2^H \cos\pi/4 + U_2^V \sin\pi/4 = 0$

$$\underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{A} \begin{pmatrix} U_1^H \\ U_1^V \\ U_2^H \\ U_2^V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A has rank 3: 1^{st}, 2^{nd}, and 4^{th} cols are independent.

$$C_1 \begin{pmatrix} -1 \\ -1/\sqrt{2} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = 0$$

3^{rd} row: $C_3 \frac{1}{\sqrt{2}} = 0$ $\quad C_3 = 0$

1^{st} row: $-C_1 = 0$ $\quad C_1 = 0$

2^{nd} row: $-\frac{1}{\sqrt{2}} C_1 + \frac{1}{\sqrt{2}} C_2 = 0 \Rightarrow C_2 = 0.$

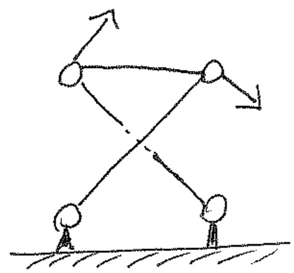
By Fund. Thm of Linear Algebra,

$$\text{rank}(A) + \dim N(A) = 4$$

$$\dim(N(A)) = 1$$

1 mode of deformation

4b)



Top bar rotates and translates.

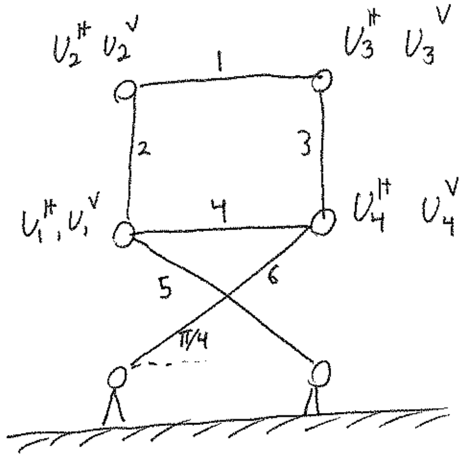
Guess $\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ in $N(A)$

Verify: $\begin{pmatrix} -1 & 0 & 1 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

c) Using $[Q,R] = qr(A^t)$, the last column of Q gives null space of A.

$\begin{pmatrix} -1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$, which is $-1/2 \times$ vector we guessed ✓

5)



Matrix constraint

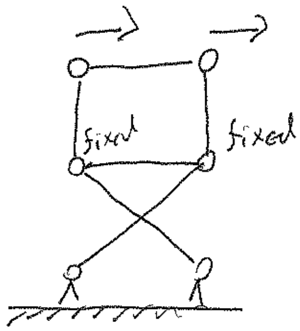
$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\cos\pi/4 & \sin\pi/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\pi/4 & \sin\pi/4 \end{pmatrix}}_{A} \begin{pmatrix} U_1^H \\ U_1^V \\ U_2^H \\ U_2^V \\ U_3^H \\ U_3^V \\ U_4^H \\ U_4^V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

A has rank 6.

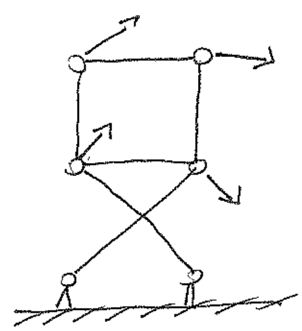
By Fund Thm of Lin Algebra, has 2 dimensional null space

2 independent modes of deformation

b)



"shear"



rotation of bar 4
square on top maintains shape

$$X = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad Ax \text{ is } 0 \ \checkmark$$

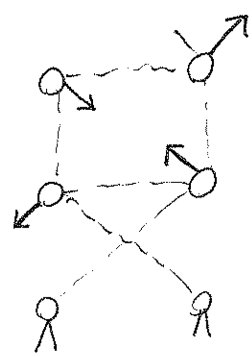
c) Last two columns of Q in

$[Q,R] = qr(A^t)$

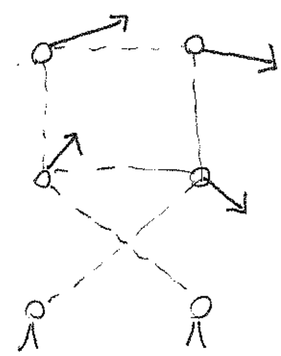
are an orthonormal basis for $N(A)$:

$$\begin{pmatrix} -0.34 \\ -0.34 \\ 0.38 \\ -0.34 \\ +0.38 \\ \cdot 0.34 \\ -0.34 \\ 0.34 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.22 \\ 0.22 \\ 0.59 \\ 0.22 \\ +0.59 \\ -0.22 \\ 0.22 \\ -0.22 \end{pmatrix}$$

Visually



and



Neither of these are our guess from (b).

But there are c_1 & c_2 s.t.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} -0.34 \\ -0.34 \\ 0.38 \\ -0.34 \\ 0.38 \\ 0.34 \\ -0.34 \\ 0.34 \end{pmatrix} + c_2 \begin{pmatrix} 0.22 \\ 0.22 \\ 0.59 \\ 0.22 \\ 0.59 \\ -0.22 \\ 0.22 \\ -0.22 \end{pmatrix}$$

$c_1 = 0.7760$

$c_2 = 1.18.$