CS G140
Graduate Computer Graphics

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Spring 2009

Lecture 6 – February 11, 2009

Today’s Topics

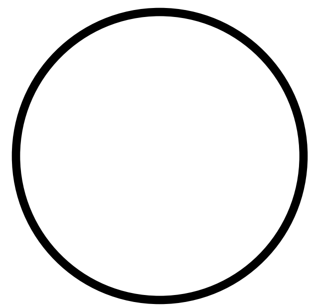
- Bezier Curves and Splines

- Parametric Bicubic Surfaces
- Quadrics

Curves

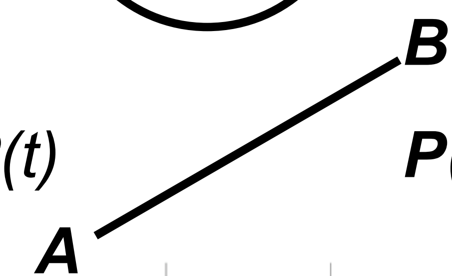
A *curve* is the continuous image of an interval in *n*-space.

Implicit $f(x, y) = 0$



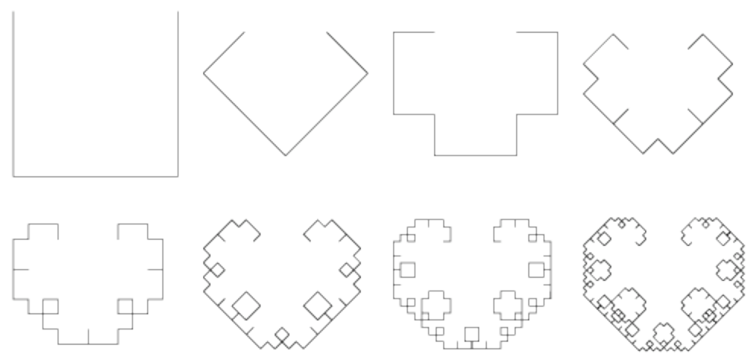
$x^2 + y^2 - R^2 = 0$

Parametric $(x(t), y(t)) = \mathbf{P}(t)$



$\mathbf{P}(t) = t\mathbf{A} + (1-t)\mathbf{B}$

Generative $proc \rightarrow (x, y)$

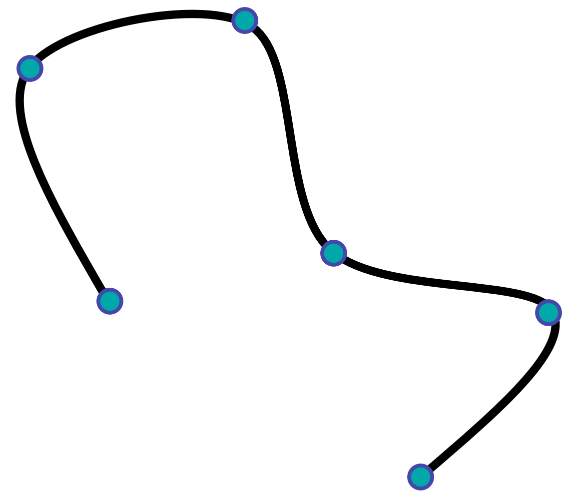


Curve Fitting

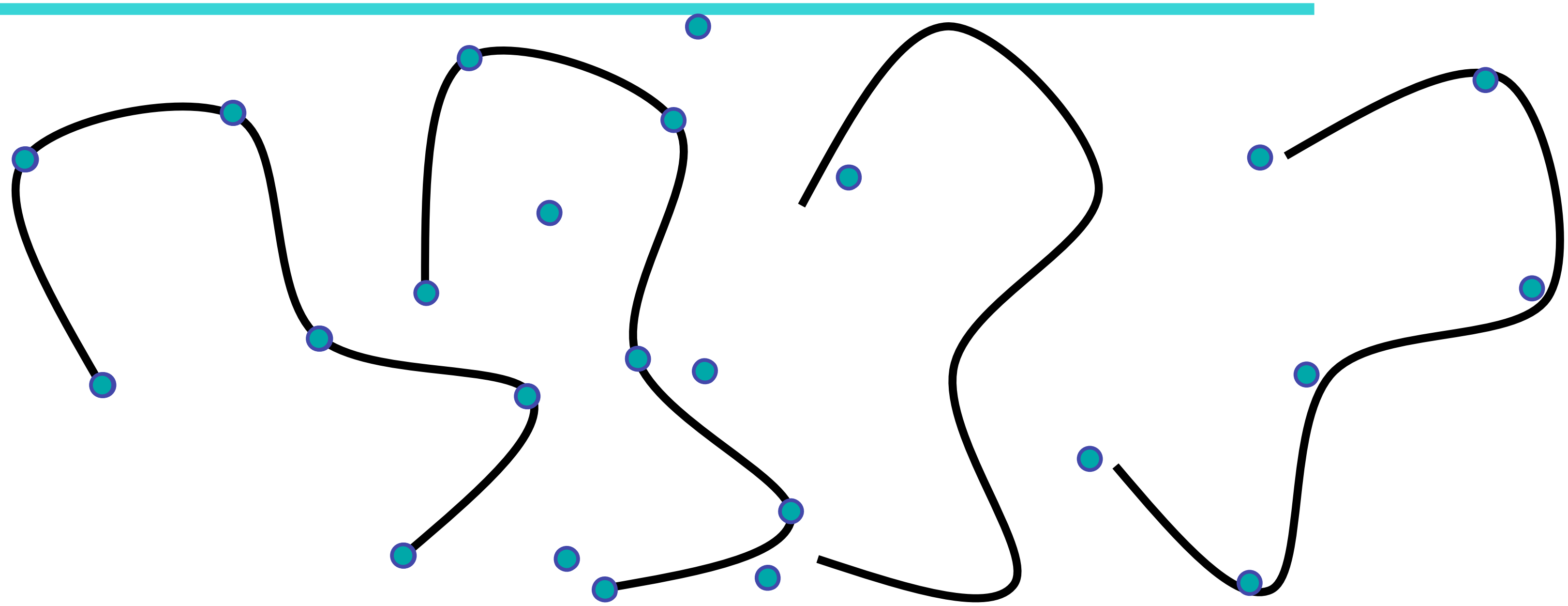
We want a curve that passes through control points.

How do we create a good curve?

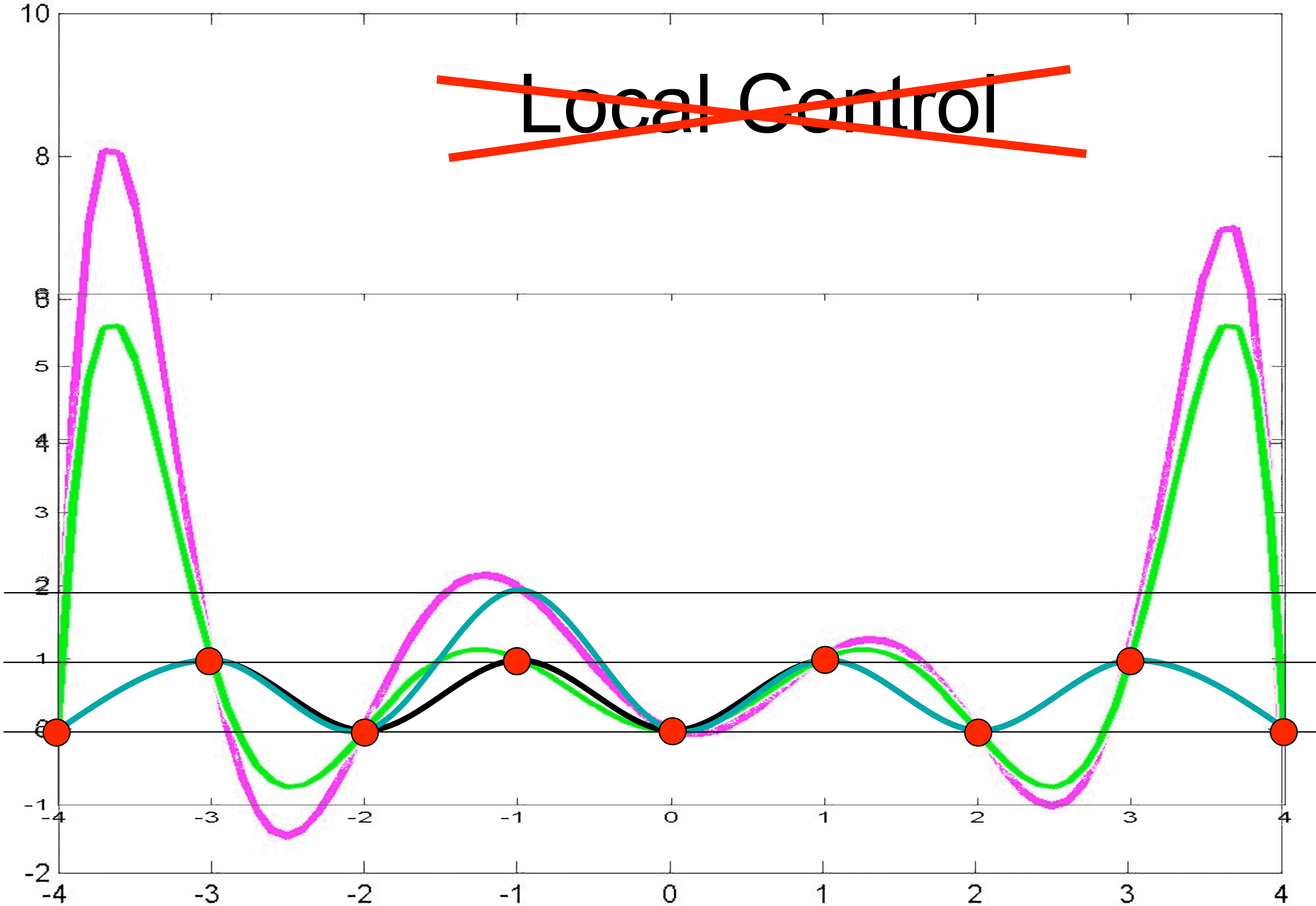
What makes a good curve?



Axis Independence

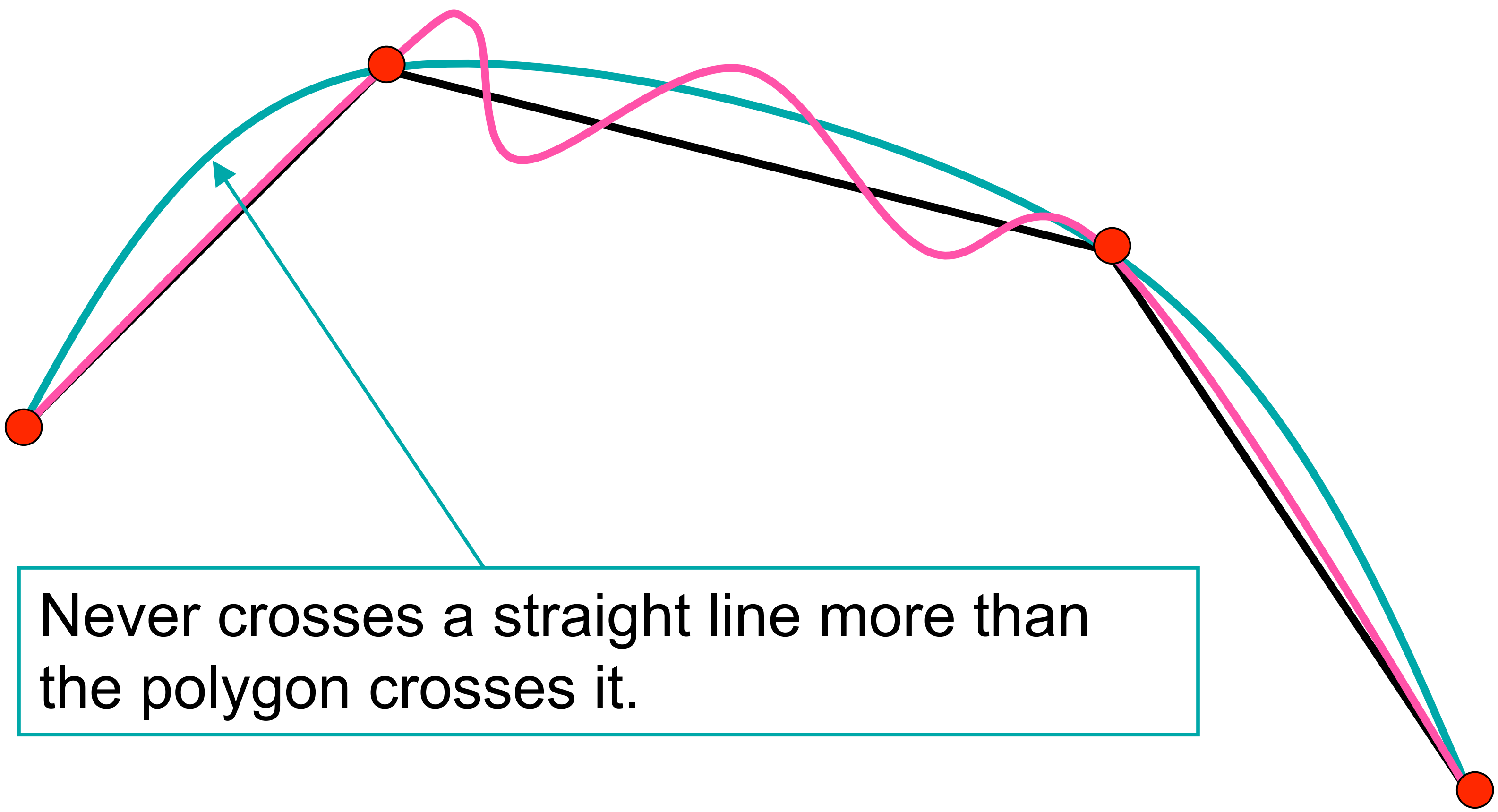


If we rotate the set of control points, we should get the rotated curve.

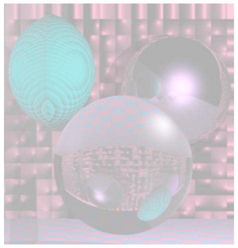


Local Control
10
8
6
5
4
3
2
1
0
-1
-2
-4
-3
-2
-1
0
1
2
3
4

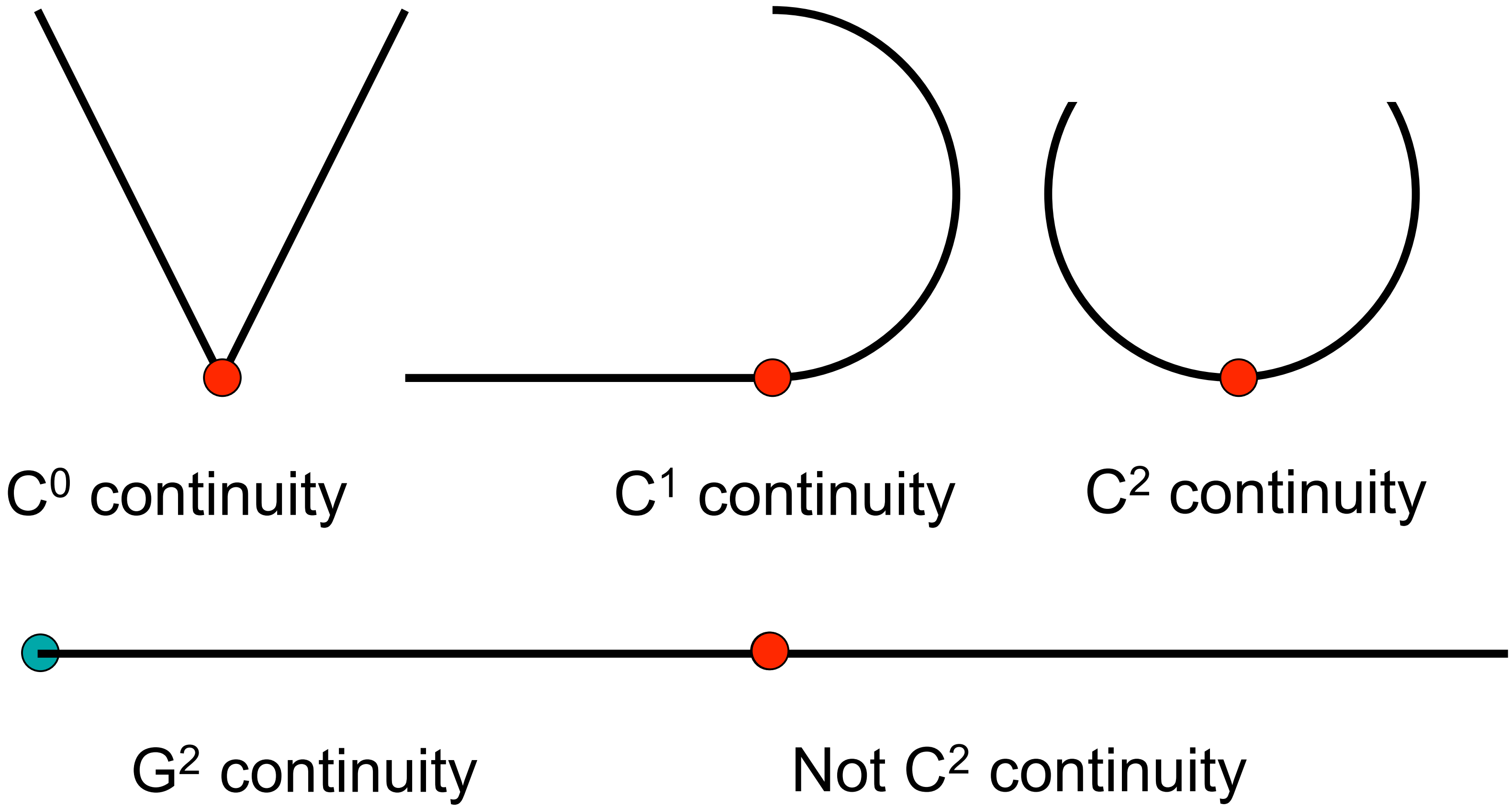
~~Variation Diminishing~~



Never crosses a straight line more than the polygon crosses it.



Continuity


C^0 continuity
C^1 continuity
C^2 continuity
G^2 continuity
Not C^2 continuity

How do we Fit Curves?

The *Lagrange interpolating polynomial* is the polynomial of degree $n\text{-}1$ that passes through the n points,

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

and is given by

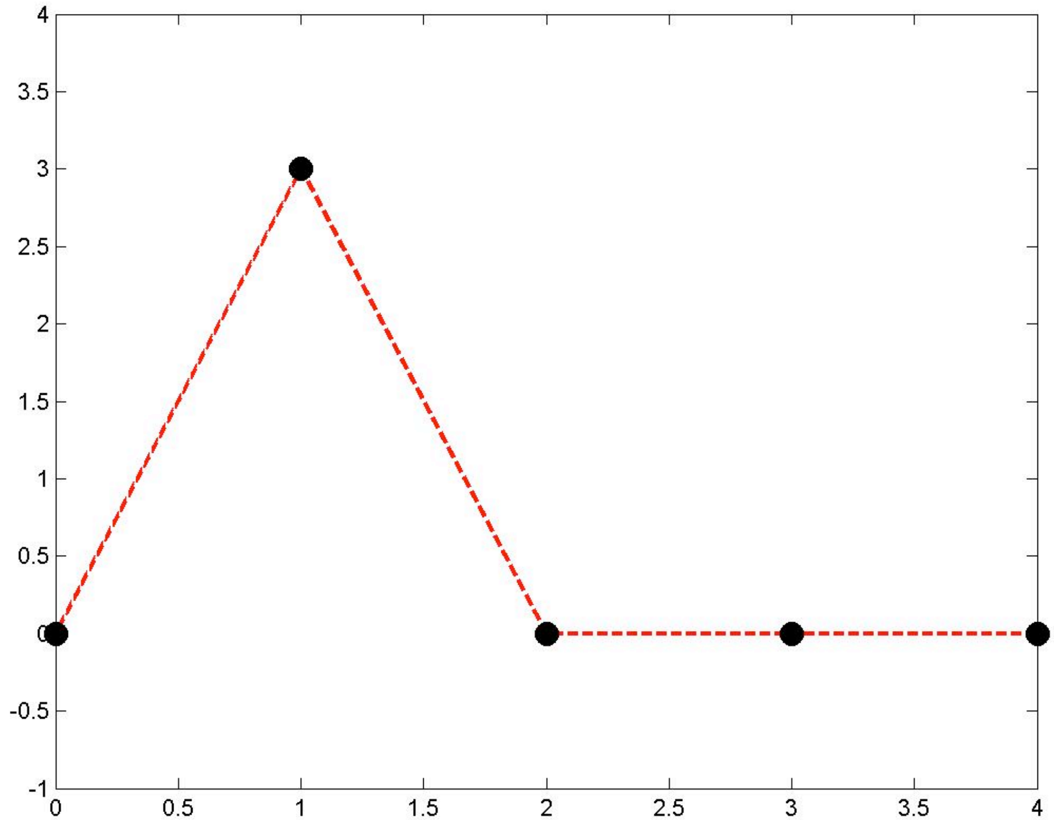
$$\mathrm{P}(\mathrm{x}) = y_1 \frac{(x - x_2)\cdots(x - x_n)}{(x_1 - x_2)\cdots(x_1 - x_n)} + y_2 \frac{(x - x_1)(x - x_3)\cdots(x - x_n)}{(x_2 - x_1)(x_2 - x_3)\cdots(x_2 - x_n)} + \cdots$$

$$+ y_n \frac{(x - x_1)\cdots(x - x_n)}{(x_n - x_1)\cdots(x_1 - x_n)}$$

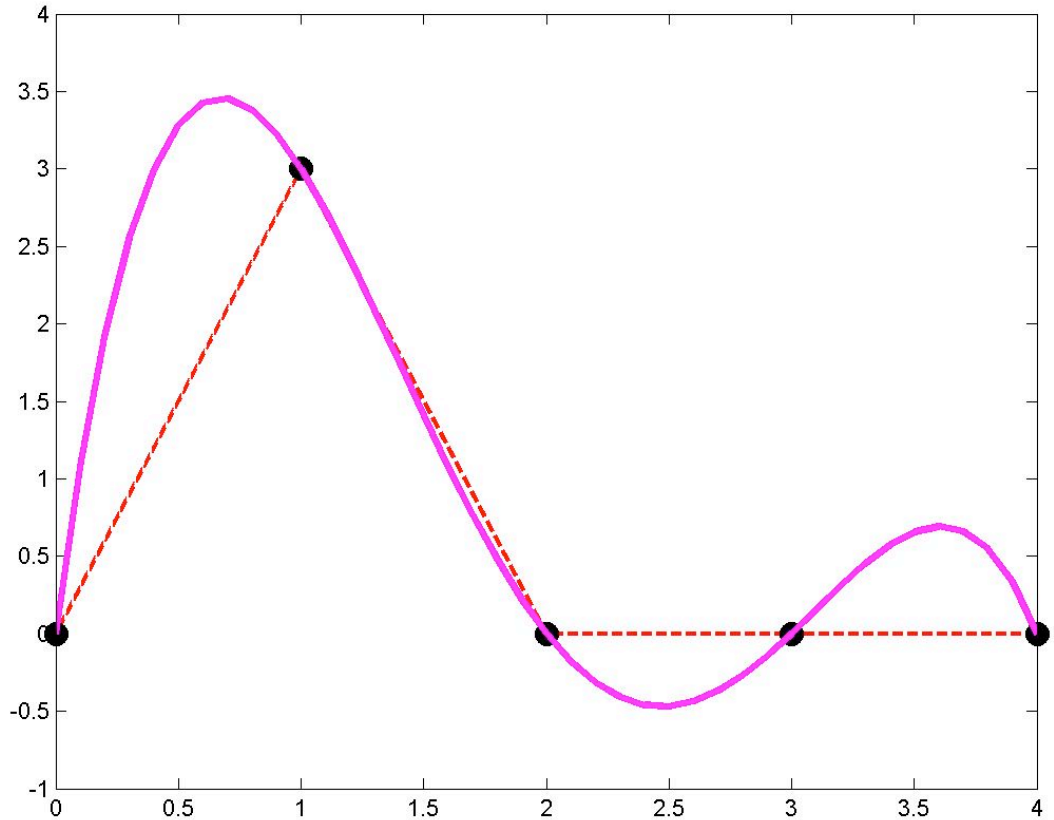
$$= \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)}$$

Lagrange Interpolating Polynomial from mathworld

Example 1

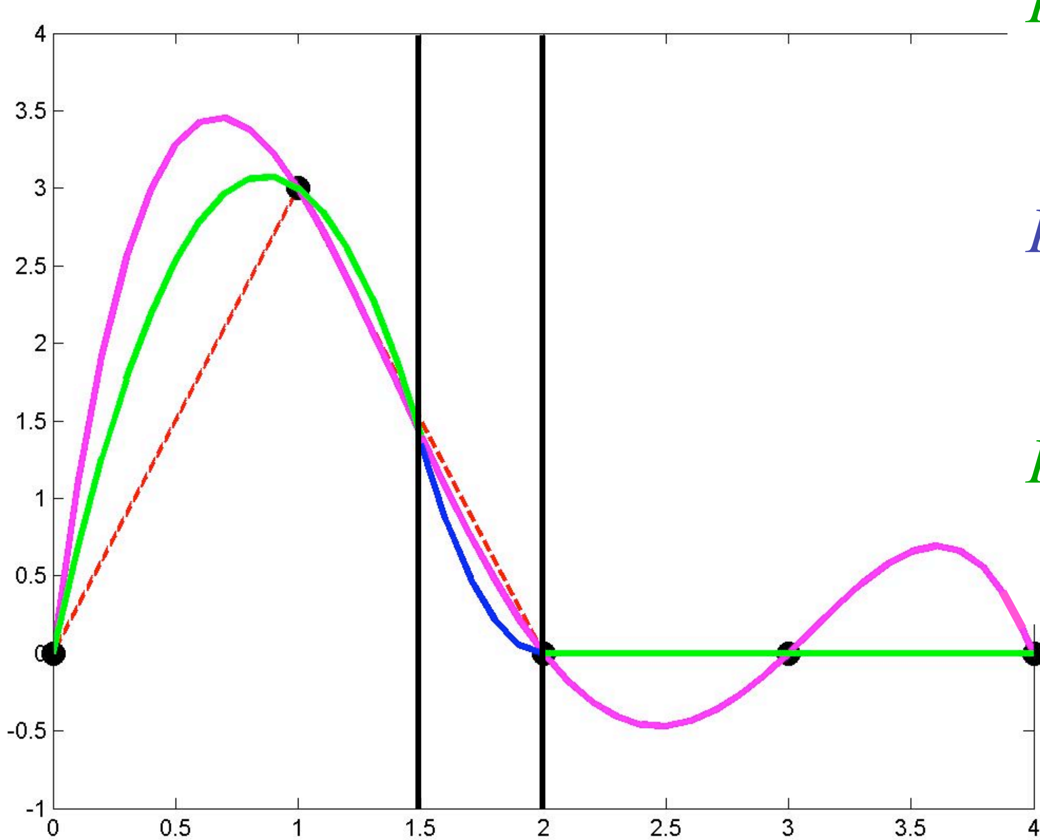


Polynomial Fit



$$P(x) = -.5x(x-2)(x-3)(x-4)$$

Piecewise Fit



$P_a(x) = 4.1249\, x\, (x - 1.7273)$

$0 \le x \le 1.5$

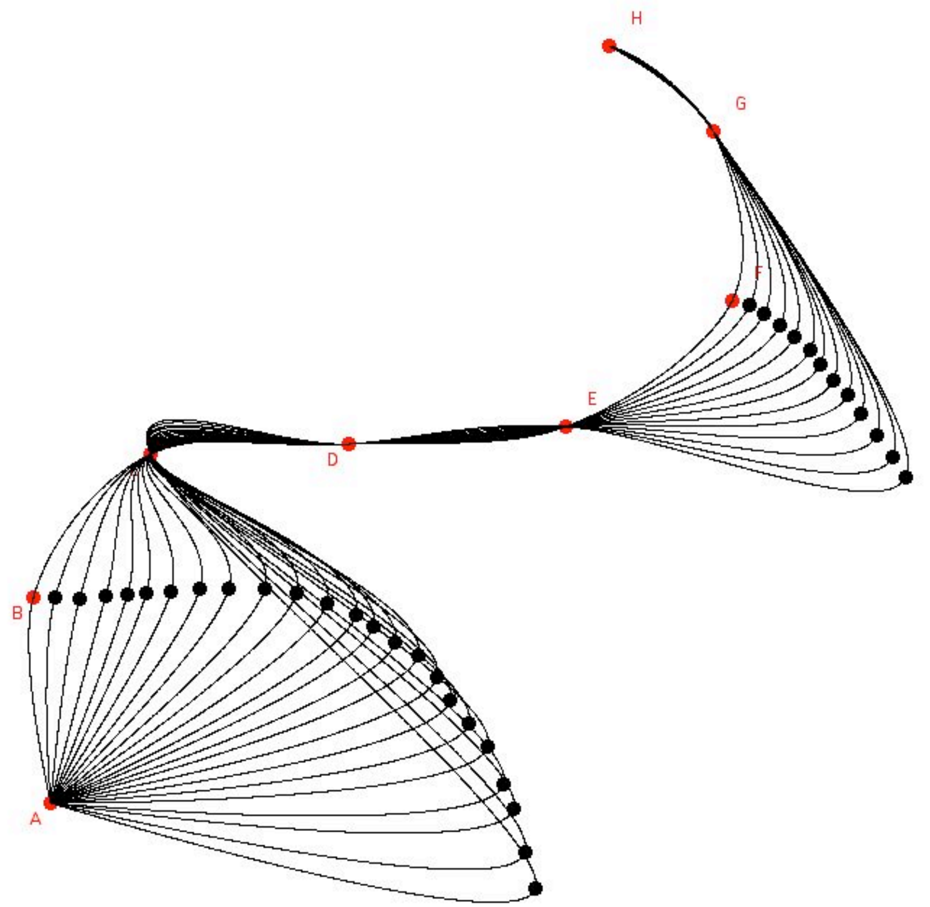
$P_b(x) = 5.4\, x\, (x - 1.7273)$

$1.5 \le x \le 2$

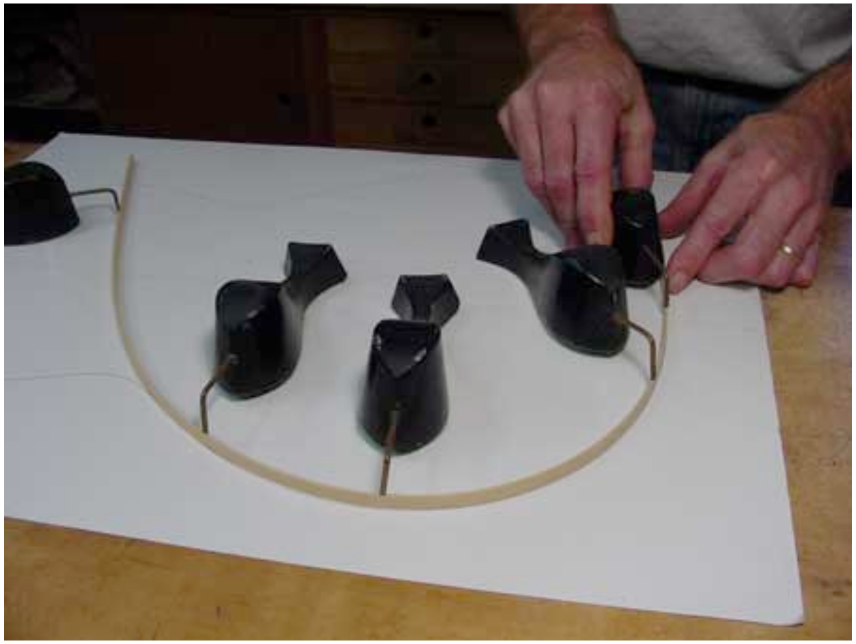
$P_c(x) = 0$

$2 \le x \le 4$

Spline Curves

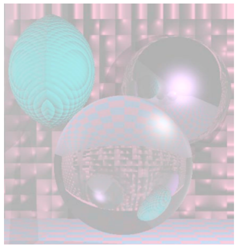


Splines and Spline Ducks

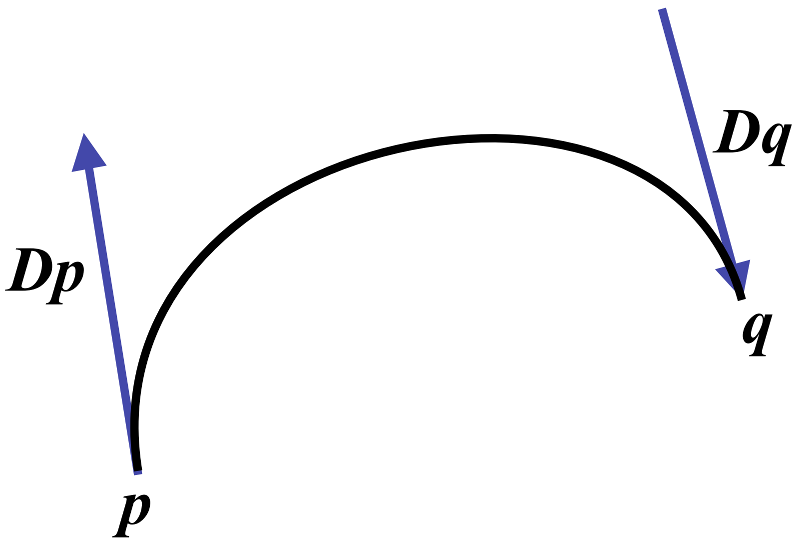


Marine Drafting Weights
http://www.frets.com/FRETSPages/Luthier/TipsTricks/DraftingWeights/draftweights.html

Drawing Spline Today (esc)



Hermite Cubics



$$\boldsymbol{P}(t) = at^3 + bt^2 + ct + d$$

$$\boldsymbol{P}(0) = \boldsymbol{p}$$

$$\boldsymbol{P}(1) = \boldsymbol{q}$$

$$\boldsymbol{P'}(0) = \boldsymbol{Dp}$$

$$\boldsymbol{P'}(1) = \boldsymbol{Dq}$$

Hermite Coefficients

$\boldsymbol{P}(t) = at^3 + bt^2 + ct + d$

$\boldsymbol{P}(0) = \boldsymbol{p}$

$\boldsymbol{P}(1) = \boldsymbol{q}$

$\boldsymbol{P}'(0) = \boldsymbol{Dp}$

$\boldsymbol{P}'(1) = \boldsymbol{Dq}$

$$\boldsymbol{P}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\boldsymbol{P}'(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

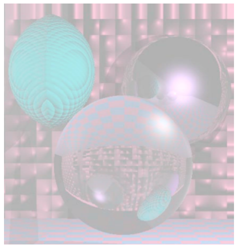
For each coordinate, we have 4 linear equations in 4 unknowns

Boundary Constraint Matrix

$$\mathbf{P}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\mathbf{P'}(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{Dp} \\ \mathbf{Dq} \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$



Hermite Matrix

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{M_H}} \underbrace{\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{Dp} \\ \boldsymbol{Dq} \end{bmatrix}}_{\boldsymbol{G_H}}$$

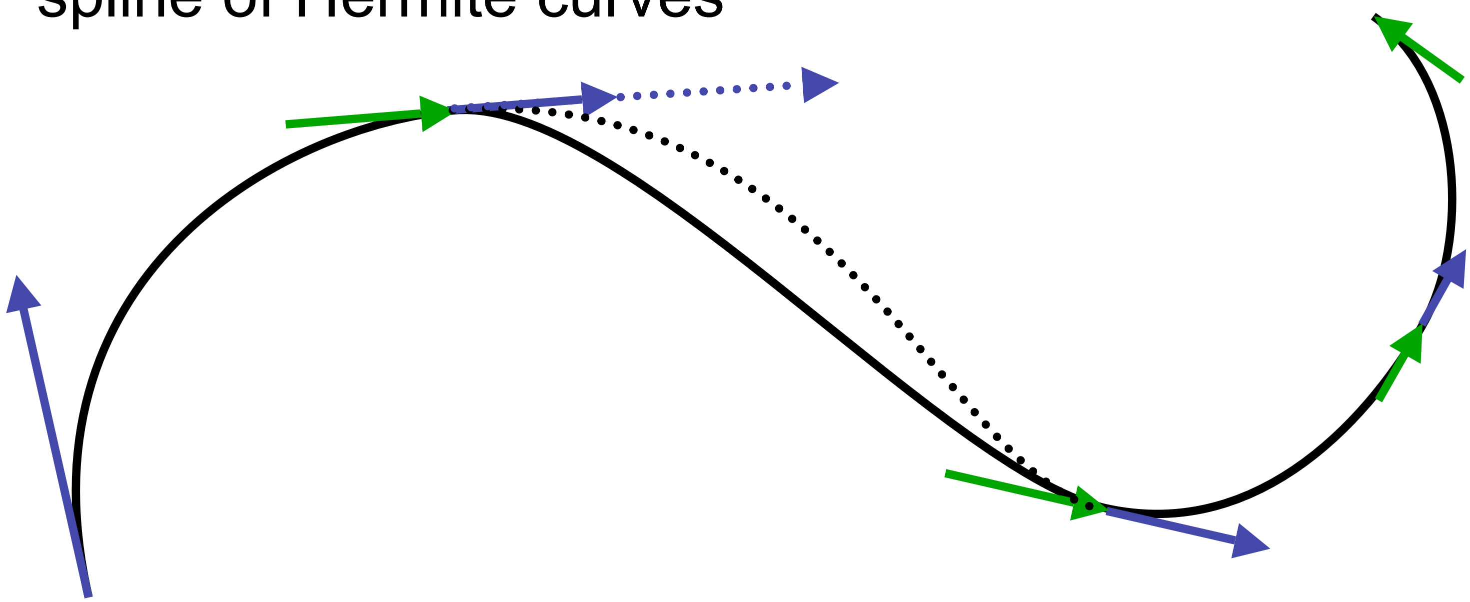
Hermite Blending Functions

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} M_H \begin{bmatrix} p \\ q \\ Dp \\ Dq \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ Dp \\ Dq \end{bmatrix}$$

$$P(t) = p \qquad + q \qquad) + Dp \qquad + Dq$$

Splines of Hermite Cubics

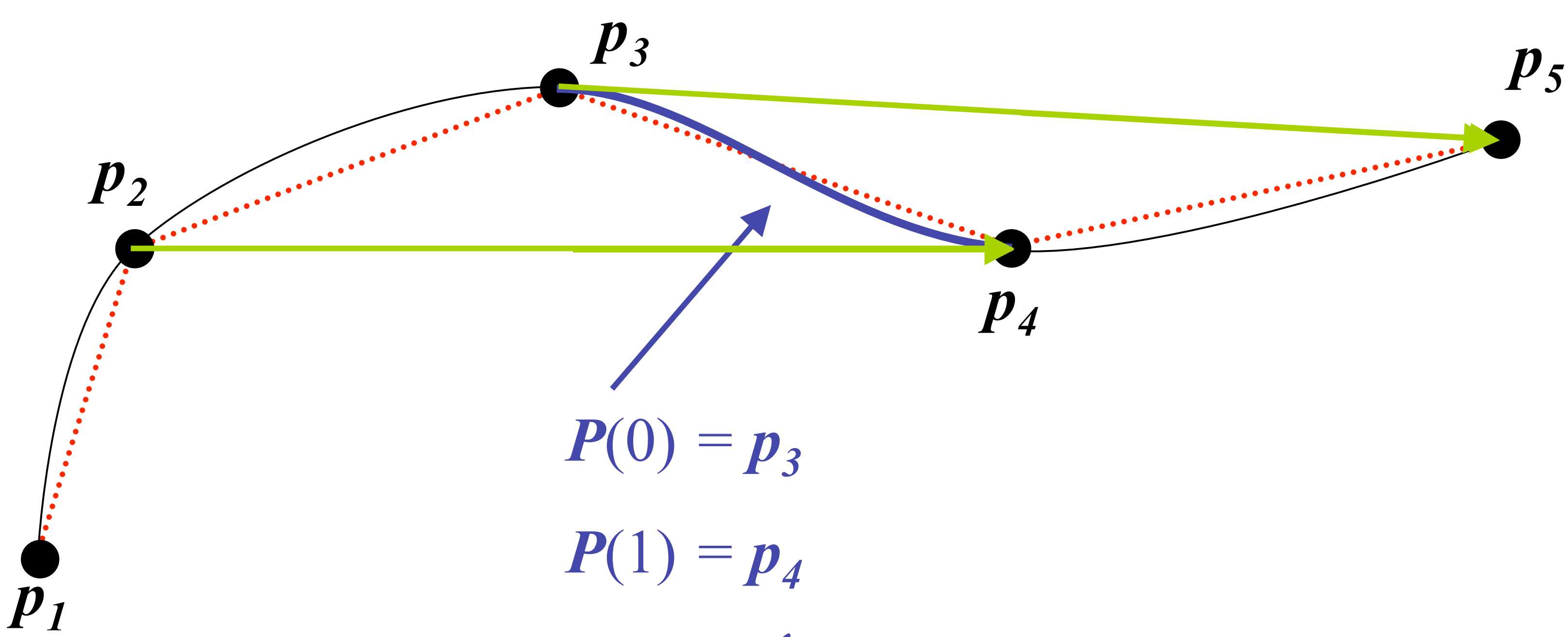
a C^1 spline of Hermite curves



a G^1 but not C^1 spline of Hermite curves

The vectors shown are 1/3 the length of the tangent vectors.

Computing the Tangent Vectors Catmull-Rom Spline



$$P(0) = p_3$$

$$P(1) = p_4$$

$$P'(0) = \tfrac{1}{2}(p_4 - p_2)$$

$$P'(1) = \tfrac{1}{2}(p_5 - p_3)$$

Cardinal Spline

The Catmull-Rom spline

$\boldsymbol{P}(0) = \boldsymbol{p}_3$

$\boldsymbol{P}(1) = \boldsymbol{p}_4$

$\boldsymbol{P}'(0) = \frac{1}{2}(\boldsymbol{p}_4 - \boldsymbol{p}_2)$

$\boldsymbol{P}'(1) = \frac{1}{2}(\boldsymbol{p}_5 - \boldsymbol{p}_3)$

is a special case of the Cardinal spline

$\boldsymbol{P}(0) = \boldsymbol{p}_3$

$\boldsymbol{P}(1) = \boldsymbol{p}_4$

$\boldsymbol{P}'(0) = (1 - t)(\boldsymbol{p}_4 - \boldsymbol{p}_2)$

$\boldsymbol{P}'(1) = (1 - t)(\boldsymbol{p}_5 - \boldsymbol{p}_3)$

$0 \le t \le 1$ is the *tension*.

Drawing Hermite Cubics

$$\boldsymbol{P}(t) = \boldsymbol{p}\left(2t^3 - 3t^2 + 1\right) + \boldsymbol{q}\left(-2t^3 + 3t^2\right) + \boldsymbol{Dp}\left(t^3 - 2t^2 + t\right) + \boldsymbol{Dq}\left(t^3 - t^2\right)$$

- How many points should we draw?
- Will the points be evenly distributed if we use a constant increment on t ?

- We actually draw Bezier cubics.

General Bezier Curves

Given $n+1$ control points $\boldsymbol{p}_i$

$$\boldsymbol{B}(t) = \sum_{k=0}^{n} \binom{n}{k} \boldsymbol{p_k} (1-t)^{n-k}\, t^k \qquad 0 \le t \le 1$$

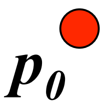
where

$$b_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k} \qquad k = 0, \cdots n$$

$$b_{k,n}(t) = (1-t)\, b_{k,n-1}(t) + t\, b_{k-1,n-1}(t) \quad 0 \le k < n$$

We will only use cubic Bezier curves, $n = 3$.

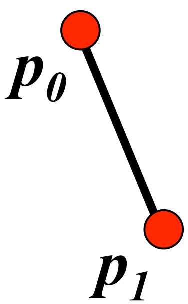
Low Order Bezier Curves



$n = 0$

$$b_{0,0}(t) = 1$$

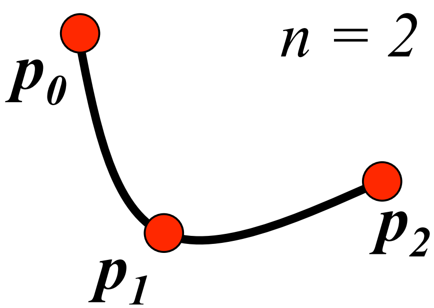
$$\boldsymbol{B}(t) = \boldsymbol{p_0}\, b_{0,0}(t) = \boldsymbol{p_0} \qquad 0 \le t \le 1$$



$n = 1$

$$b_{0,1}(t) = 1 - t \qquad b_{1,1}(t) = t$$

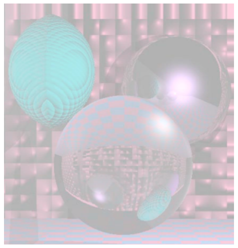
$$\boldsymbol{B}(t) = (1 - t)\, \boldsymbol{p_0} + t\, \boldsymbol{p_1} \qquad 0 \le t \le 1$$



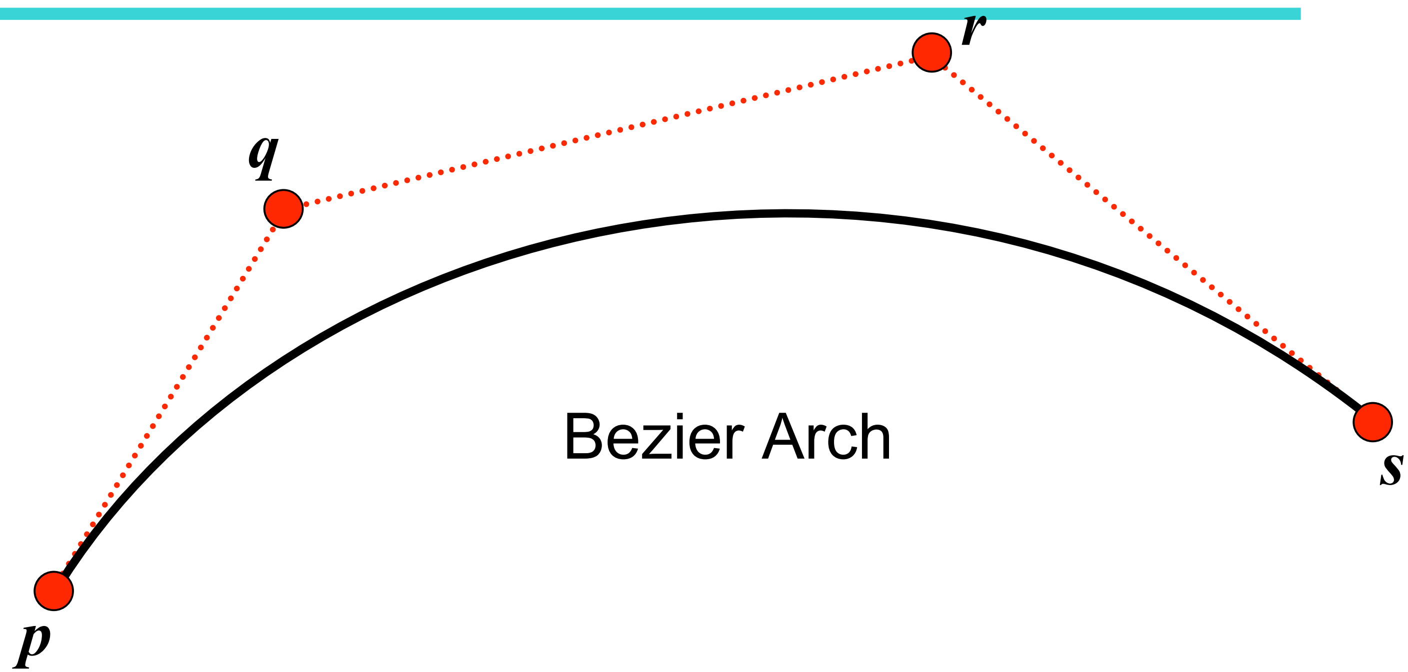
$n = 2$

$$b_{0,2}(t) = (1 - t)^2 \qquad b_{1,2}(t) = 2t\,(1 - t) \qquad b_{2,2}(t) = t^2$$

$$\boldsymbol{B}(t) = (1 - t)^2\, \boldsymbol{p_0} + 2t\,(1 - t)\boldsymbol{p_1} + t^2\, \boldsymbol{p_2} \qquad 0 \le t \le 1$$



Bezier Curves



$n = 3$ $\quad b_{0,3}(t) = (1 - t)^3 \quad b_{1,3}(t) = 3t\,(1 - t)^2$

$b_{2,3}(t) = 3t^2(1 - t) \quad b_{2,3}(t) = t^3$

$$\boldsymbol{B}(t) = (1 - t)^3\,\boldsymbol{p} + 3t\,(1 - t)^2\boldsymbol{q} + 3t^2(1 - t)\boldsymbol{r} + t^3\boldsymbol{s} \qquad 0 \le t \le 1$$

Bezier Matrix

$\boldsymbol{B}(t) = (1 - t)^3 \boldsymbol{p} + 3t(1 - t)^2 \boldsymbol{q} + 3t^2(1 - t)\boldsymbol{r} + t^3 \boldsymbol{s} \qquad 0 \le t \le 1$

$\boldsymbol{B}(t) = \boldsymbol{a}\,t^3 + \boldsymbol{b}t^2 + \boldsymbol{c}t + \boldsymbol{d} \qquad 0 \le t \le 1$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{M_B}} \underbrace{\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \\ \boldsymbol{s} \end{bmatrix}}_{\boldsymbol{G_B}}$$

Geometry Vector

The Hermite Geometry Vector $G_H = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{Dp} \\ \boldsymbol{Dq} \end{bmatrix}$ $\quad H(t) = TM_H G_H$

The Bezier Geometry Vector $G_B = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \\ \boldsymbol{s} \end{bmatrix}$ $\quad B(t) = TM_B G_B$

$$T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}$$

Properties of Bezier Curves

$P(0) = p$ $\qquad$ $P(1) = s$

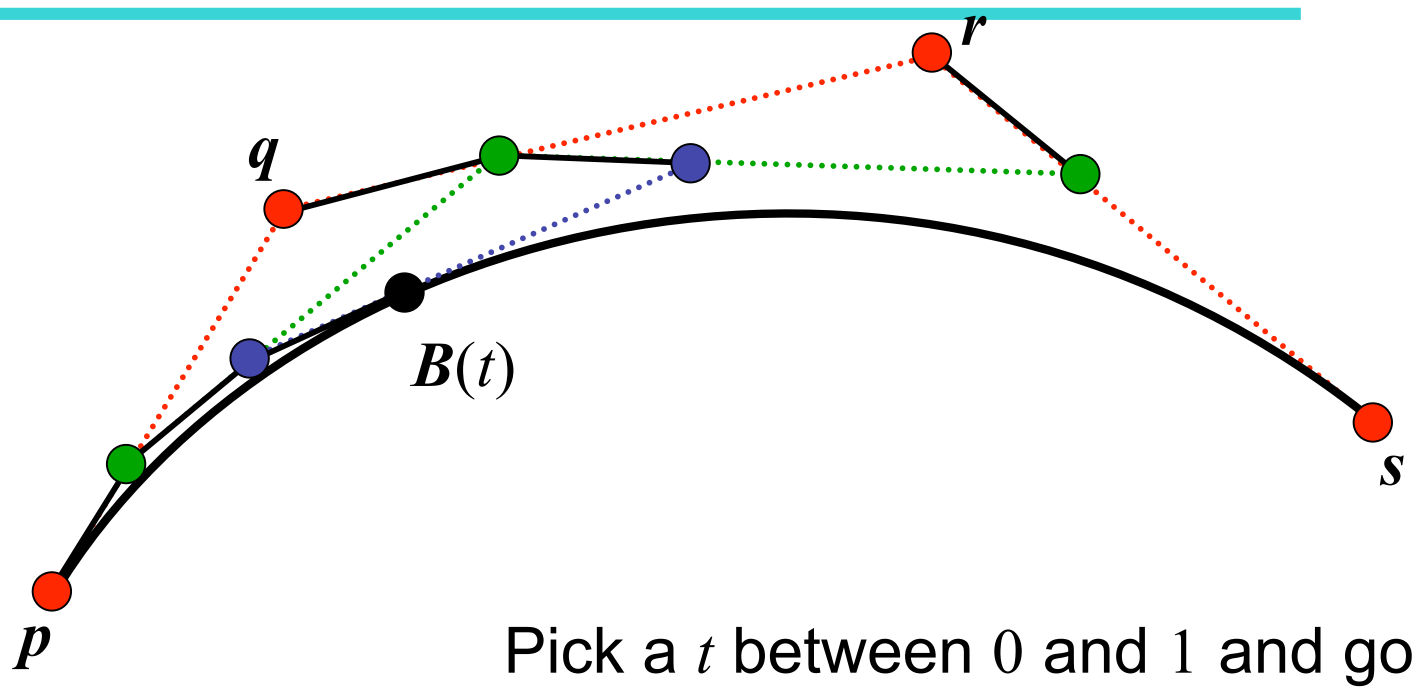
$P'(0) = 3(q - p)$ $\qquad$ $P'(1) = 3(s - r)$

The curve is tangent to the segments ***pq*** and ***rs***.

The curve lies in the convex hull of the control points since

$$\sum_{k=1}^{3} b_{k,3}(t) = \sum_{k=1}^{3} \binom{3}{k} (1-t)^k t^{3-k} = ((1-t)+t)^3 = 1$$

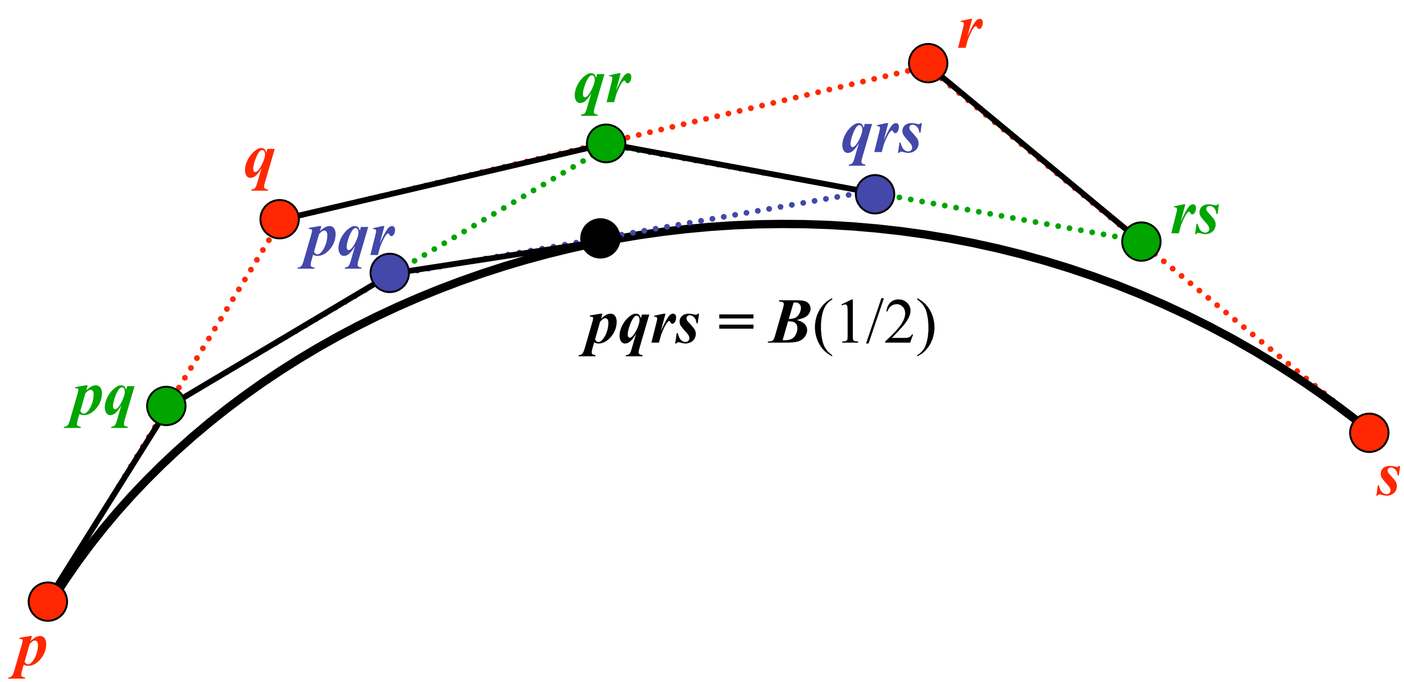
Geometry of Bezier Arches



Pick a t between 0 and 1 and go t of the way along each edge.

Join the endpoints and do it again.

Geometry of Bezier Arches



We only use $t = 1/2$.

```
drawArch(P, Q, R, S){
 if (ArchSize(P, Q, R, S) <= .5 ) Dot(P);
 else{
  PQ = (P + Q)/2;
  QR = (Q + R)/2;
  RS = (R + S)/2;

  PQR = (PQ + QR)/2;
  QRS = (QR + RS)/2;

  PQRS = (PQR + QRS)/2

  drawArch(P, PQ, PQR, PQRS);
  drawArch(PQRS, QRS, RS, S);
 }
}
```

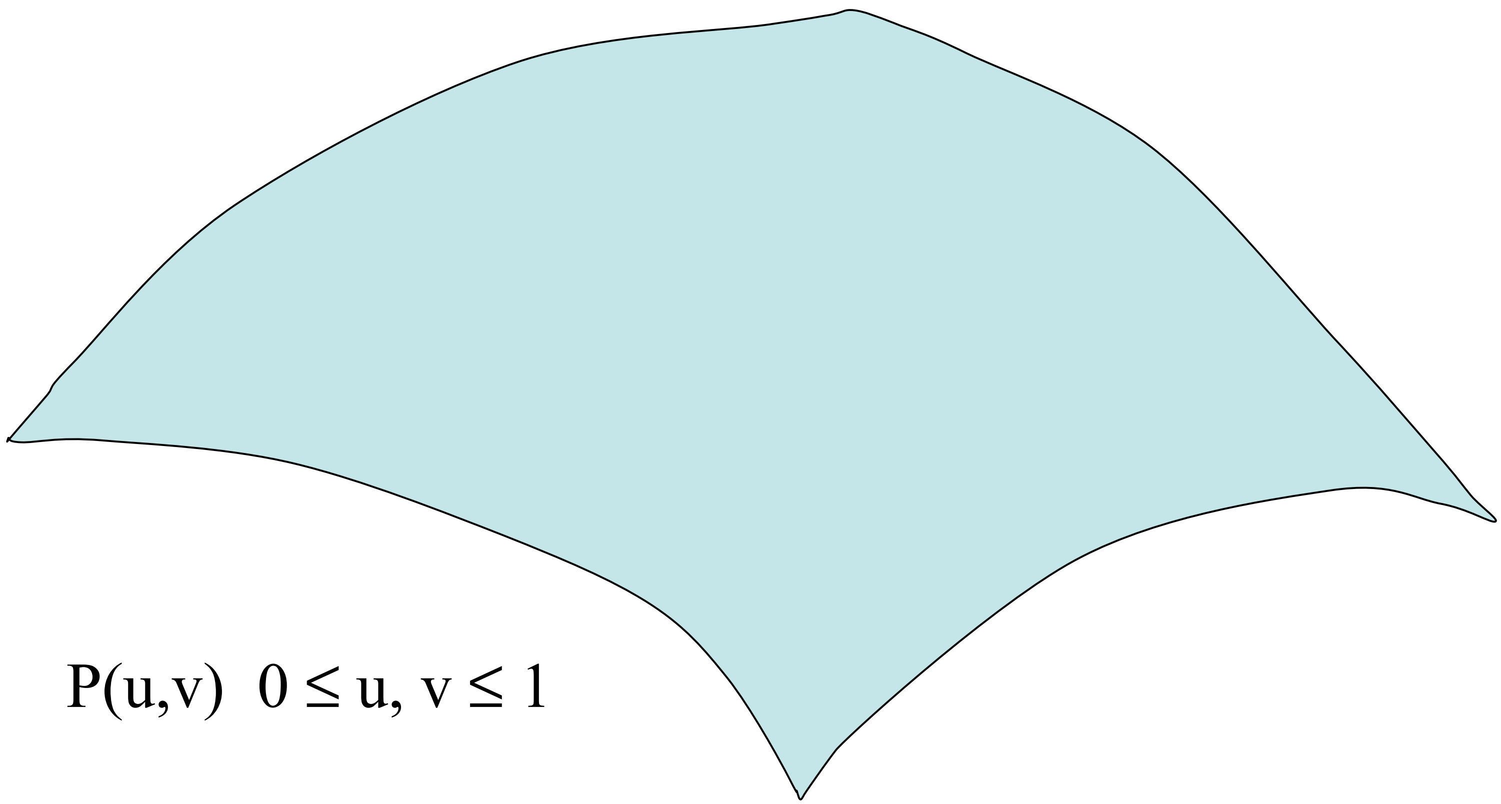
Putting it All Together

- Bezier Arches and Catmull-Rom Splines

Time for a Break

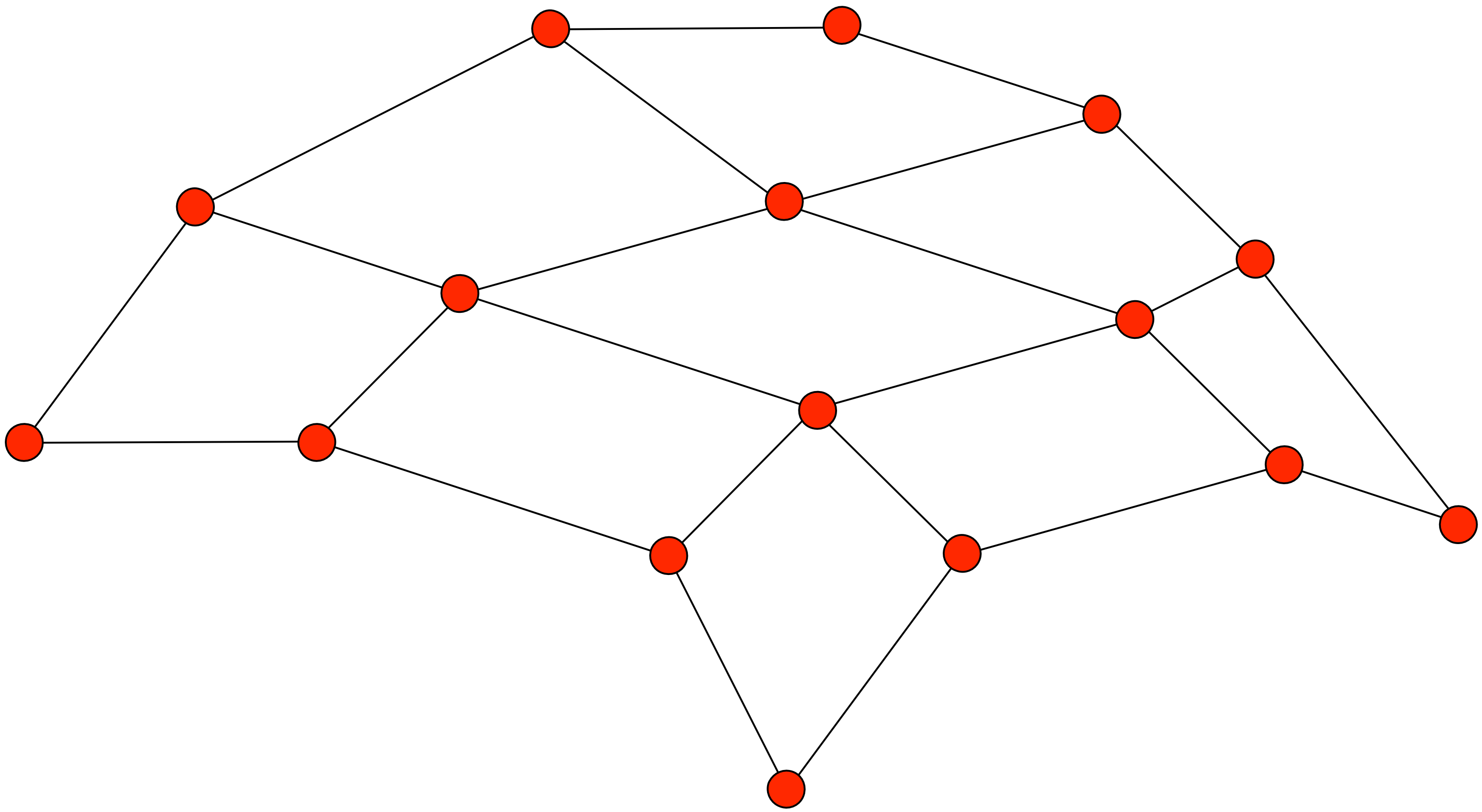
Surface Patch

A *patch* is the continuous image of a square in *n*-space.

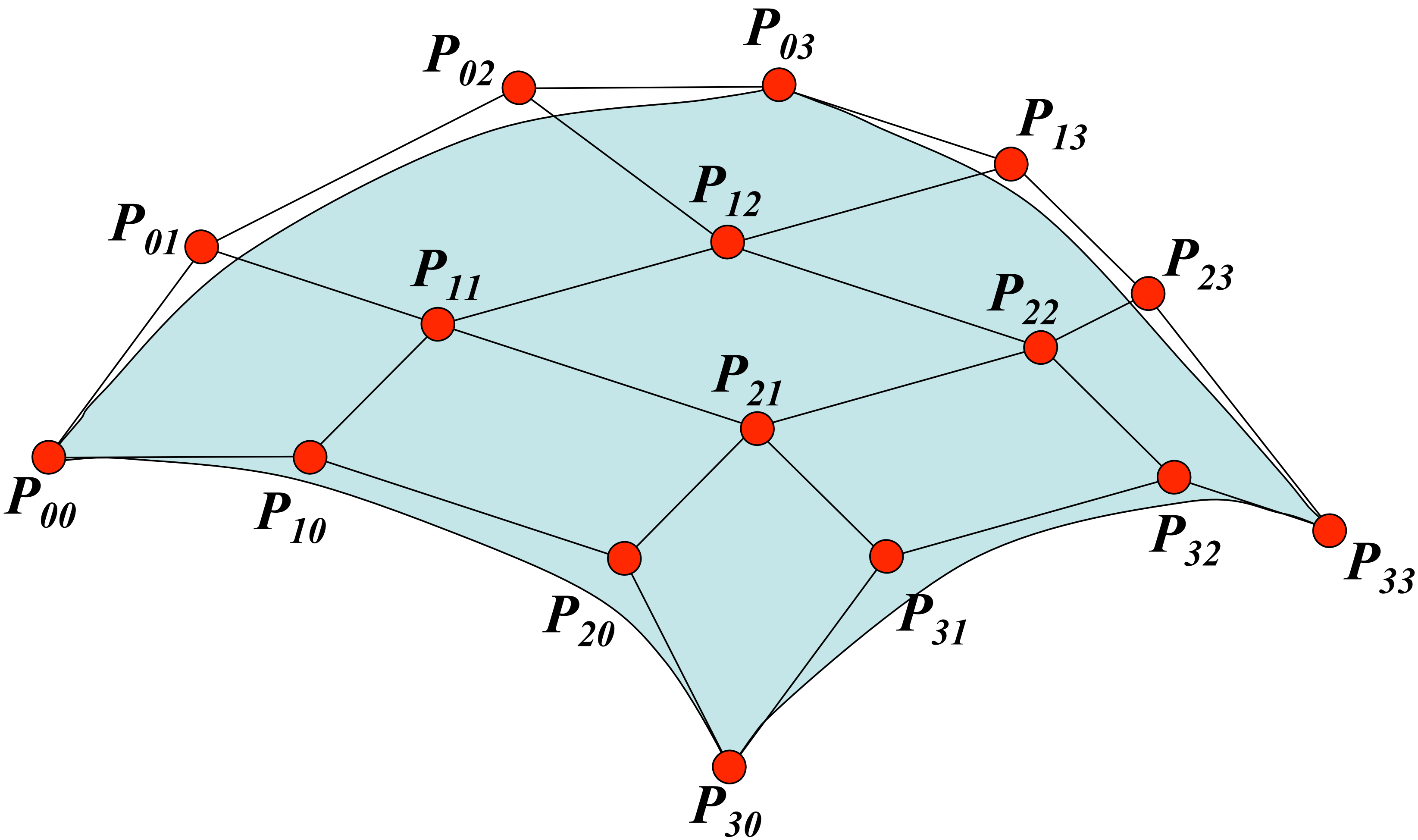


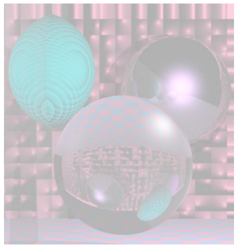
$P(u,v) \quad 0 \le u, v \le 1$

Bezier Patch Geometry



Bezier Patch



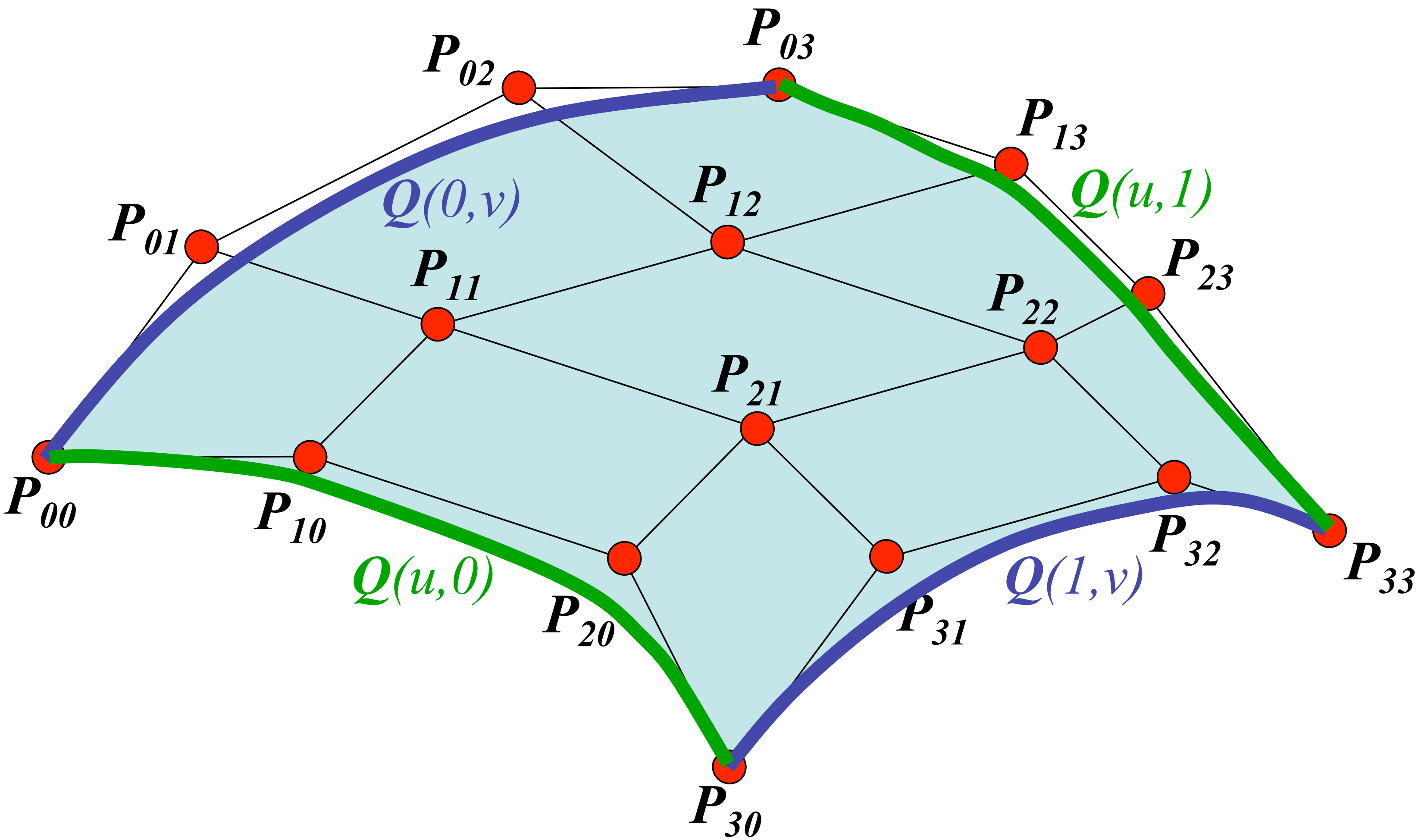


Bezier Patch Computation

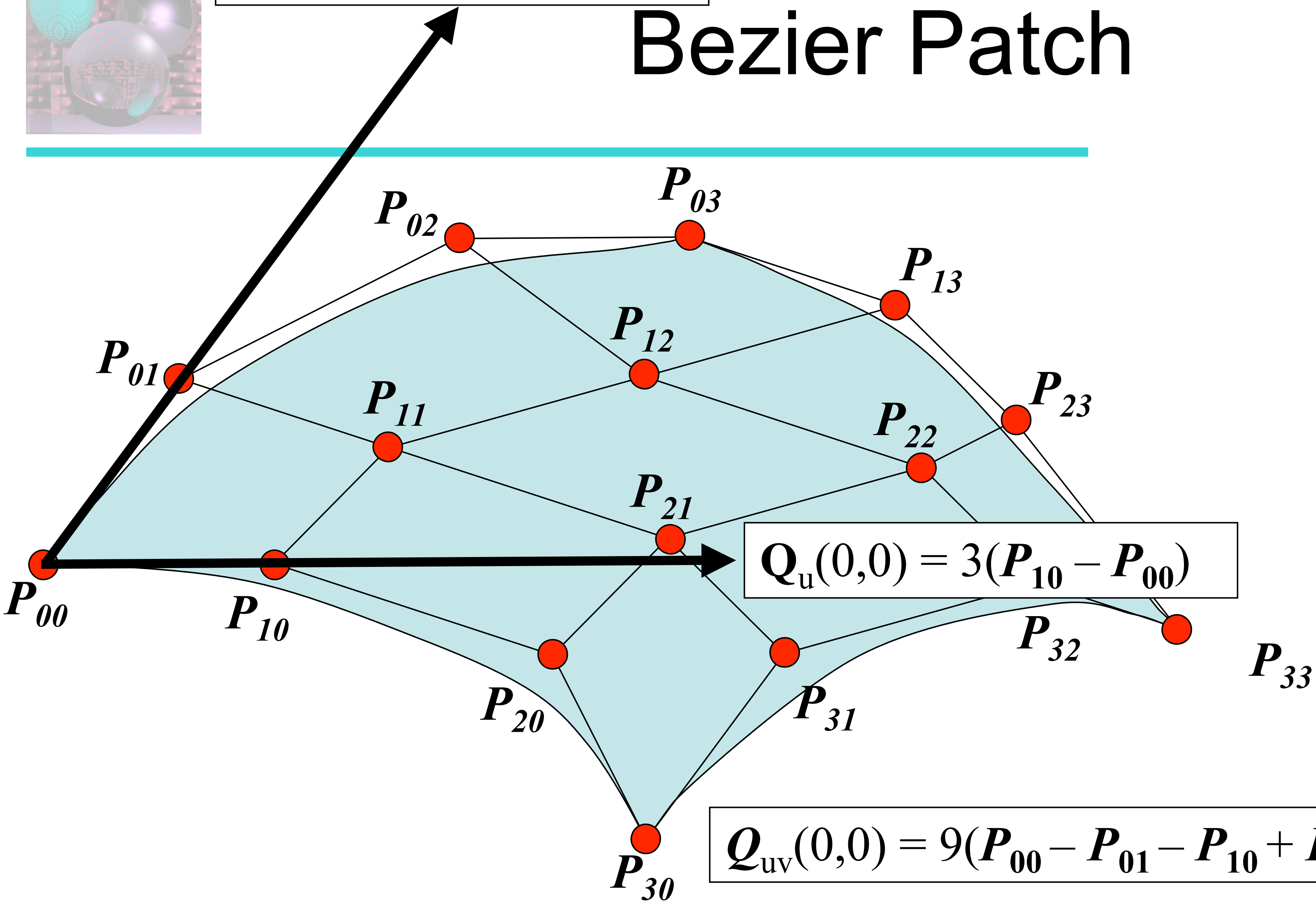
$$\boldsymbol{Q}(u,v)=\sum_{i=0}^{3}\sum_{j=0}^{3}\boldsymbol{P_{ij}}B_i(u)B_j(v)=\begin{bmatrix}u^3 & u^2 & u & 1\end{bmatrix}\begin{bmatrix}M_B\boldsymbol{P}M_B{}^T\end{bmatrix}\begin{bmatrix}v^3\\ v^2\\ v\\ 1\end{bmatrix}$$

$$M_B=M_B^T=\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\qquad \boldsymbol{P}=\begin{bmatrix}\boldsymbol{P_{00}} & \boldsymbol{P_{01}} & \boldsymbol{P_{02}} & \boldsymbol{P_{03}}\\ \boldsymbol{P_{10}} & \boldsymbol{P_{11}} & \boldsymbol{P_{12}} & \boldsymbol{P_{13}}\\ \boldsymbol{P_{20}} & \boldsymbol{P_{21}} & \boldsymbol{P_{22}} & \boldsymbol{P_{23}}\\ \boldsymbol{P_{30}} & \boldsymbol{P_{31}} & \boldsymbol{P_{32}} & \boldsymbol{P_{33}}\end{bmatrix}$$

Bezier Patch



Bezier Patch



$$\mathbf{Q}_v(0,0) = 3(\boldsymbol{P}_{01} - \boldsymbol{P}_{00})$$

$$\mathbf{Q}_u(0,0) = 3(\boldsymbol{P}_{10} - \boldsymbol{P}_{00})$$

$$\boldsymbol{Q}_{uv}(0,0) = 9(\boldsymbol{P}_{00} - \boldsymbol{P}_{01} - \boldsymbol{P}_{10} + \boldsymbol{P}_{11})$$

Properties of Bezier Surfaces

- A Bézier patch transforms in the same way as its control points under all affine transformations
- All *u* = constant and *v* = constant lines in the patch are Bézier curves.
- A Bézier patch lies completely within the convex hull of its control points.
- The corner points in the patch are the four corner control points.
- A Bézier surface does not in general pass through its other control points.

Rendering Bezier Patches with a mesh

1. Consider each row of control points as defining 4 separate Bezier curves: $Q_0(u) \ldots Q_3(u)$

$$\begin{bmatrix} Q_0(u) & Q_1(u) & Q_2(u) & Q_3(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} B\mathbf{P}$$

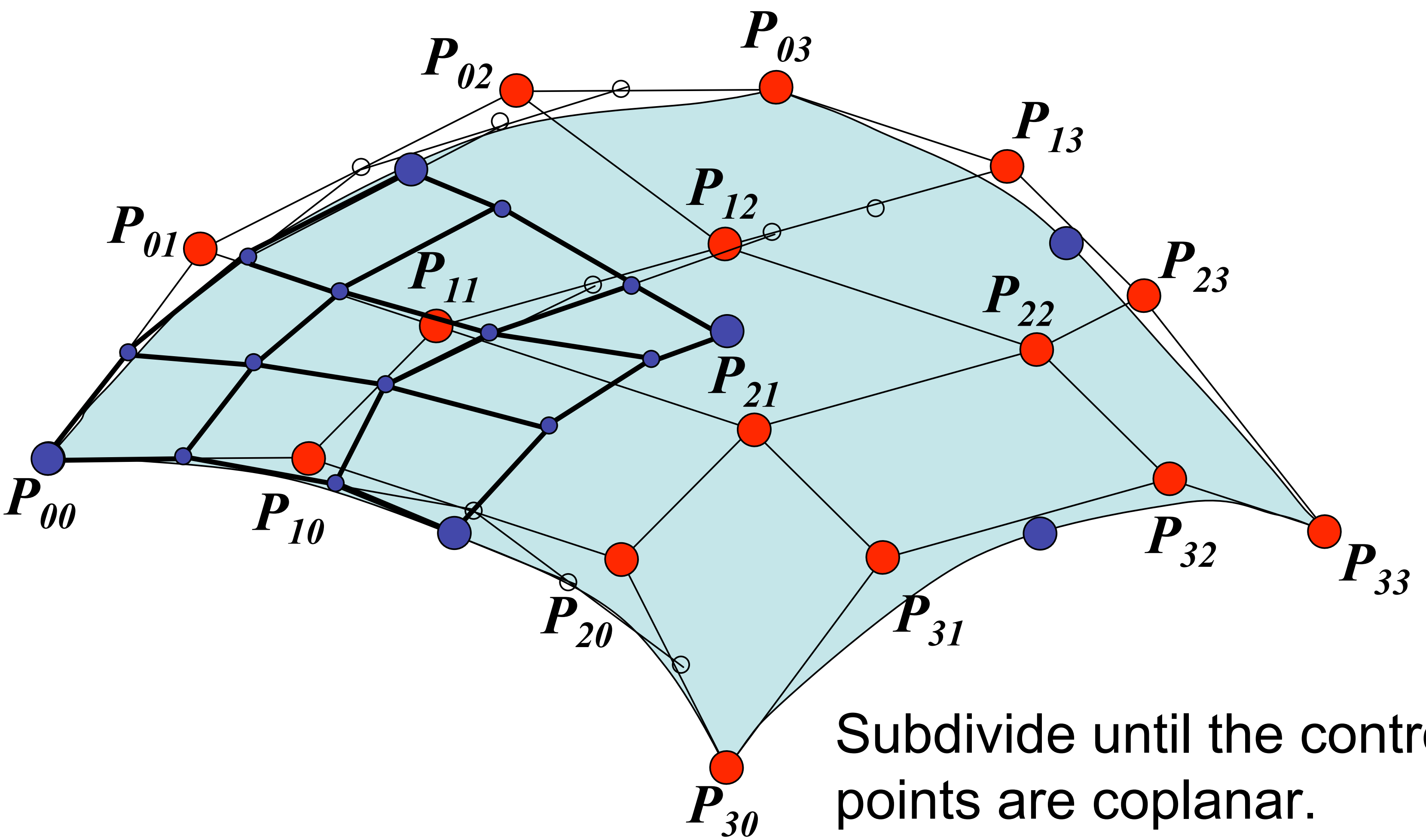
2. For some value of **u**, say 0.1, for each Bezier curve, calculate $Q_0(u) \ldots Q_3(u)$.

3. Use these derived points as the control points for new Bezier curves running in the **v** direction

4. Generate edges and polygons from grid of surface points.

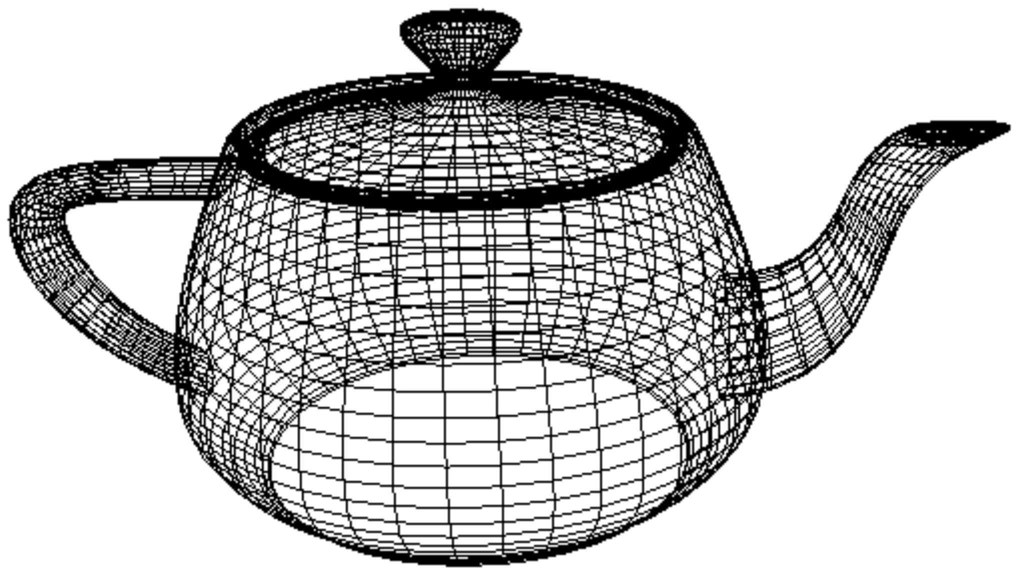
[Chris Bently - Rendering Bezier Patches]

Subdividing Bezier Patch



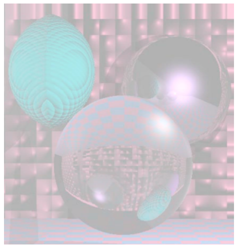
Subdivide until the control points are coplanar.

Blending Bezier Patches

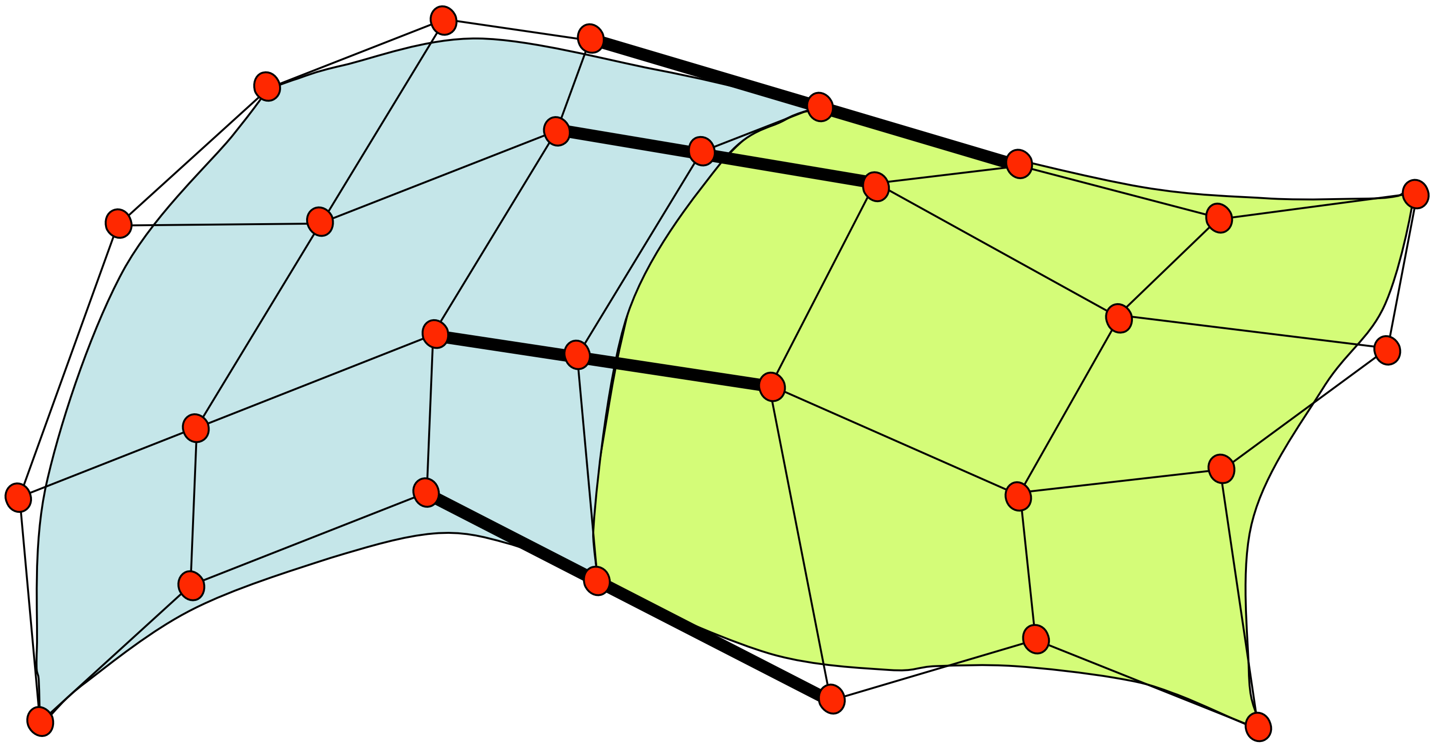


Teapot Data

```
double teapot_data[][] = {
{
    -80.00,   0.00,  30.00,      -80.00, -44.80,  30.00,
-44.80, -80.00,  30.00,        0.00, -80.00,  30.00,
    -80.00,   0.00,  12.00,      -80.00, -44.80,  12.00,
-44.80, -80.00,  12.00,        0.00, -80.00,  12.00,
    -60.00,   0.00,   3.00,      -60.00, -33.60,   3.00,
-33.60, -60.00,   3.00,          0.00,  -60.00,   3.00,
    -60.00,   0.00,   0.00,      -60.00, -33.60,   0.00,
-33.60, -60.00,   0.00,        0.00, -60.00,   0.00,
}, …
```



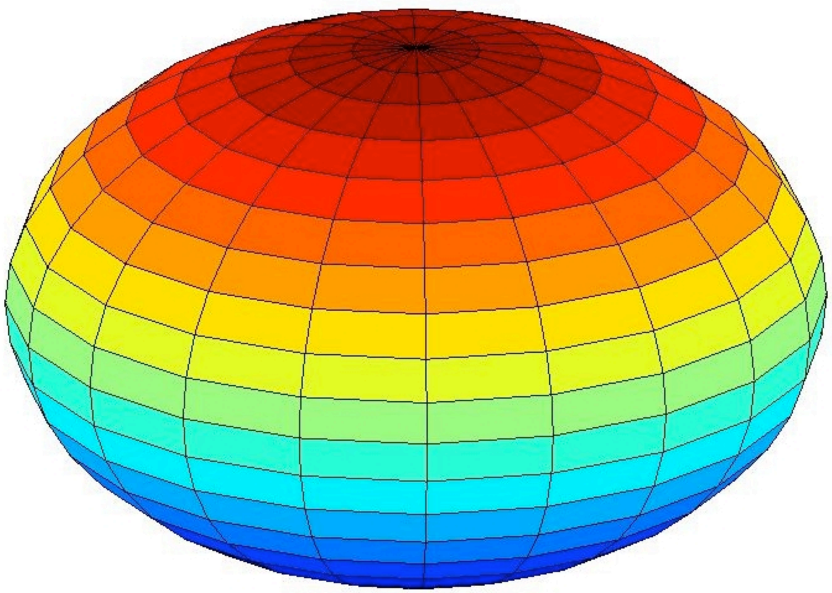
Bezier Patch Continuity



If these sets of control points are colinear, the surface will have G^1 continuity.

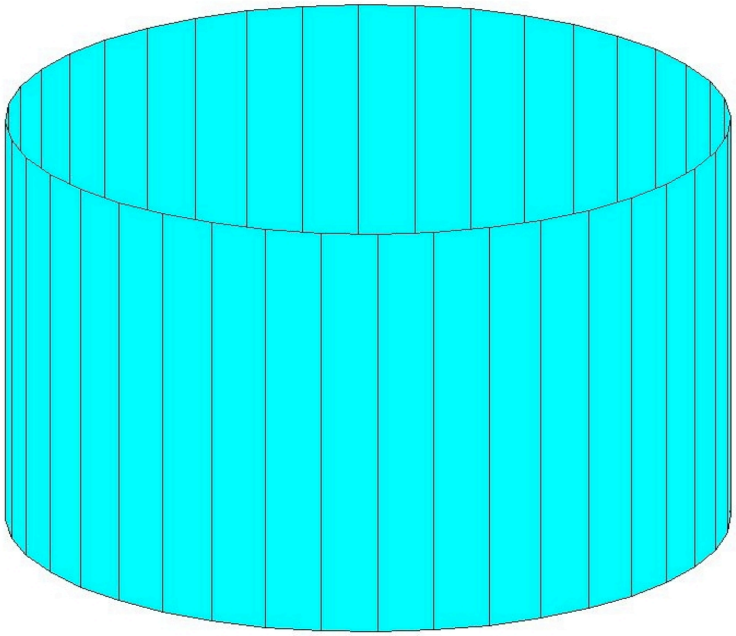
Quadric Surfaces

ellipsoid



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

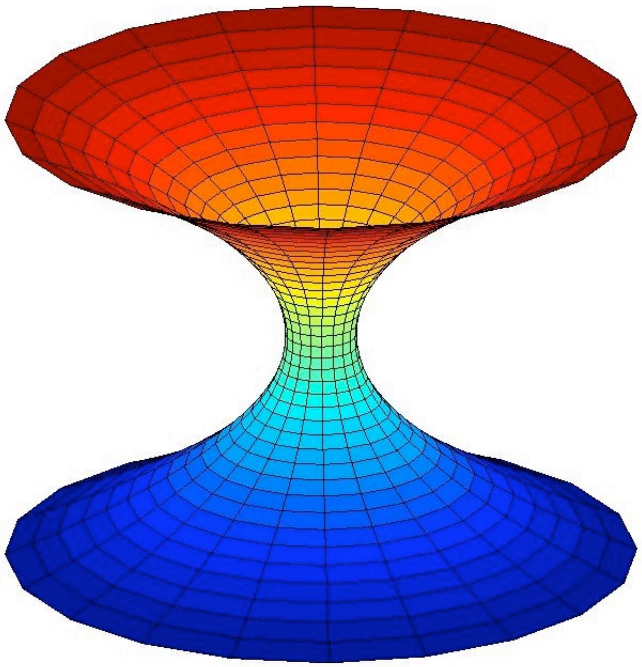
elliptic cylinder



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

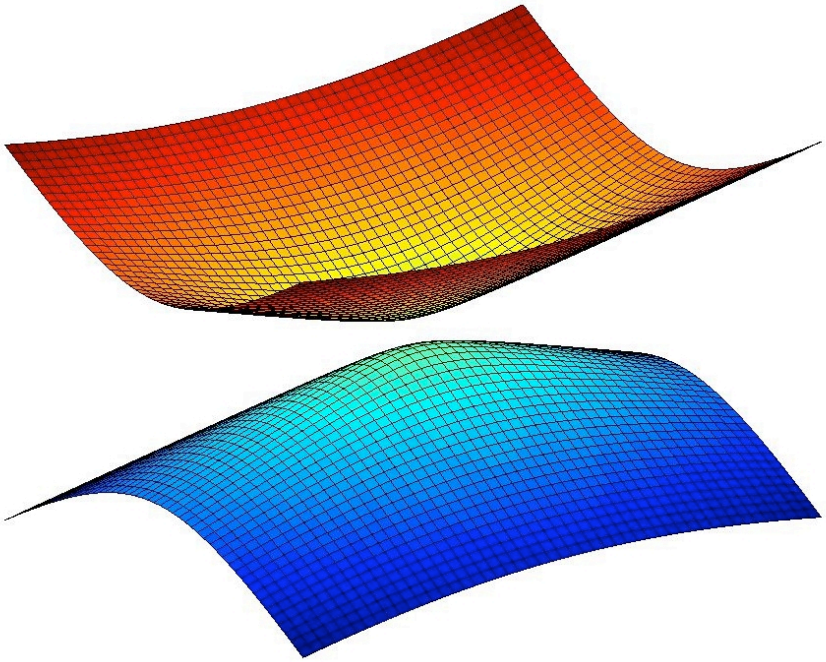
Quadric Surfaces

1-sheet hyperboloid



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

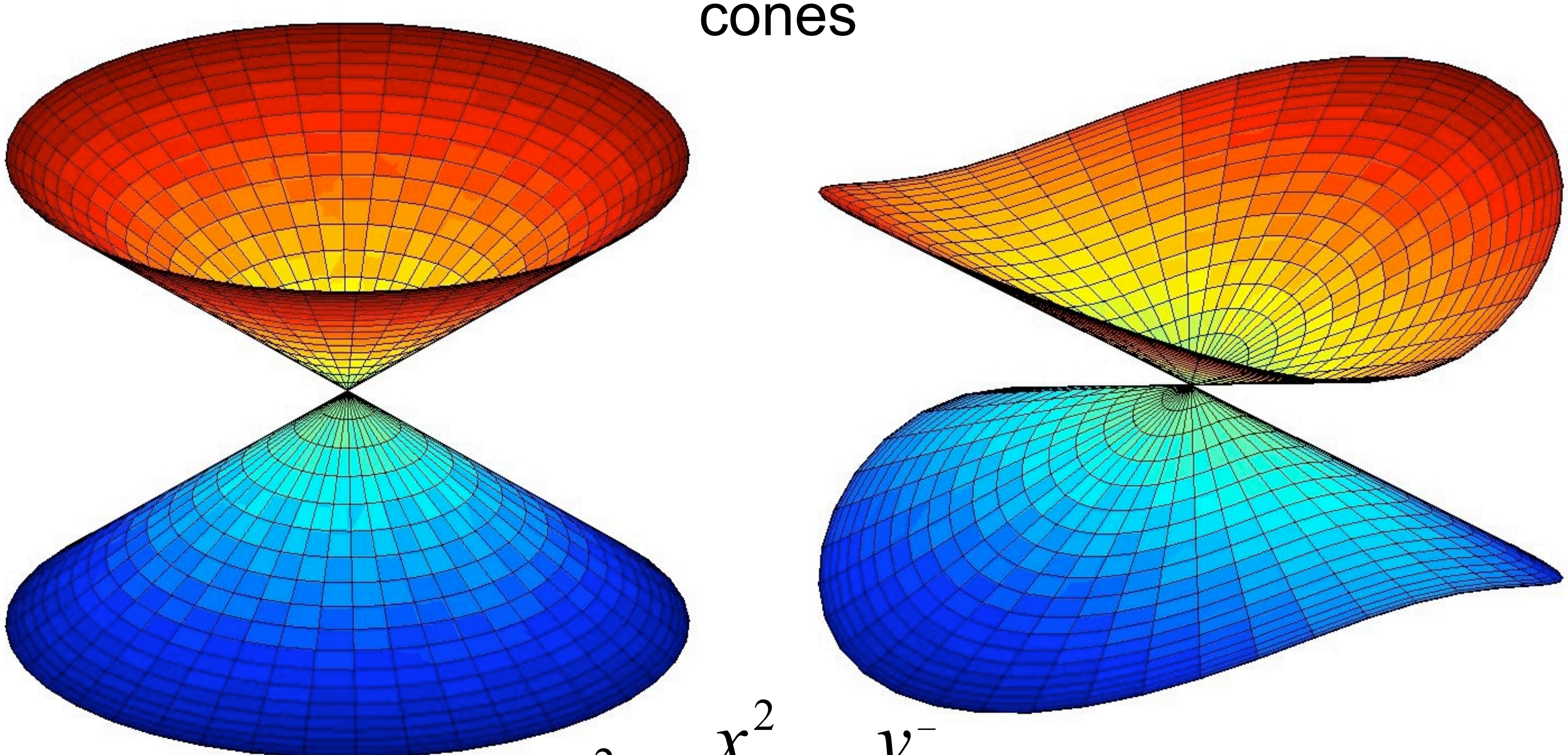
2-sheet hyperboloid



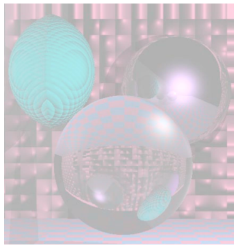
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

Quadric Surfaces

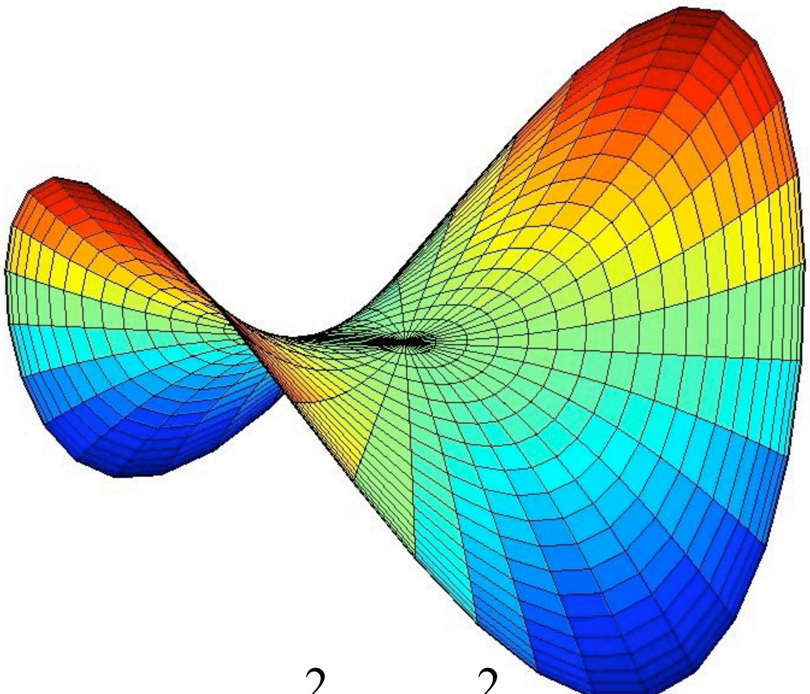
cones



$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

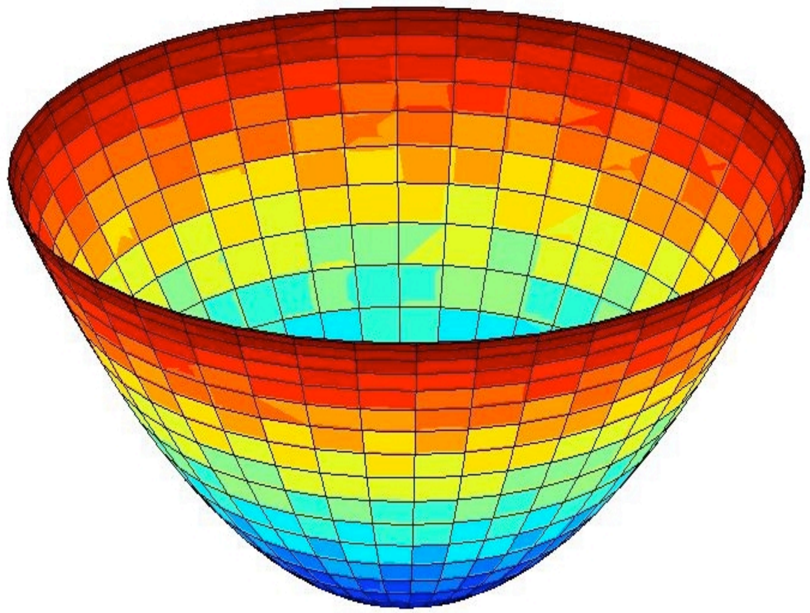


hyperbolic parabaloid



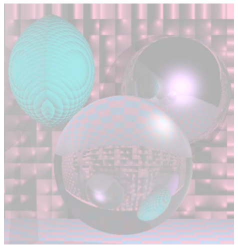
$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

elliptic parabaloid

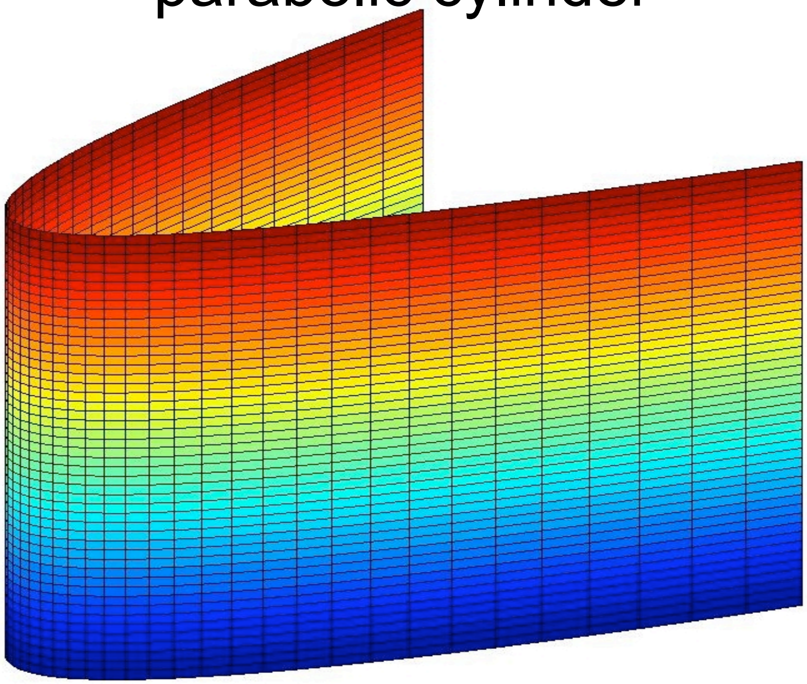


$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Quadric Surfaces

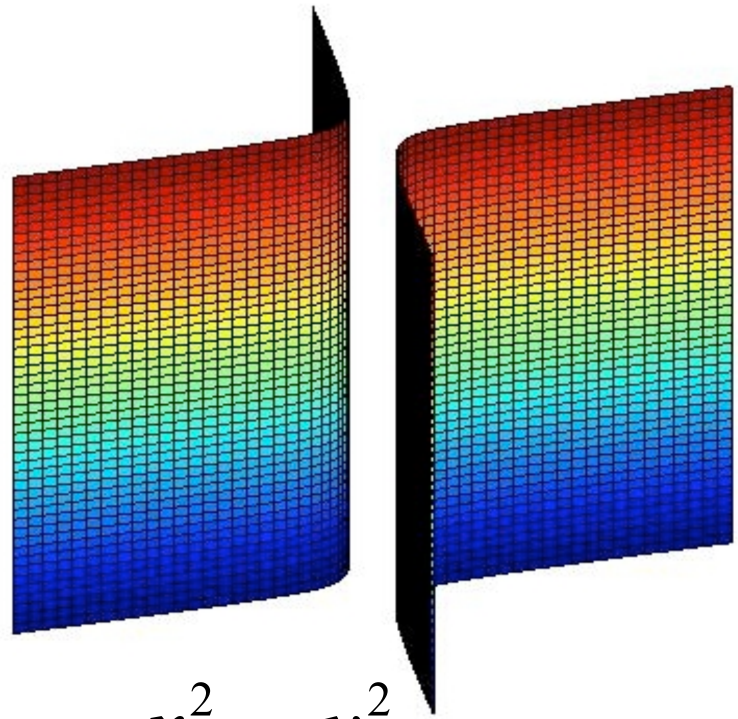


parabolic cylinder



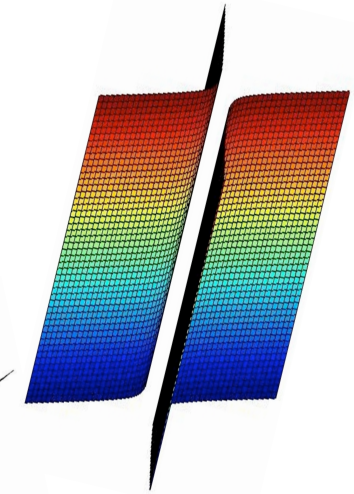
$$x^2 + 2y = 0$$

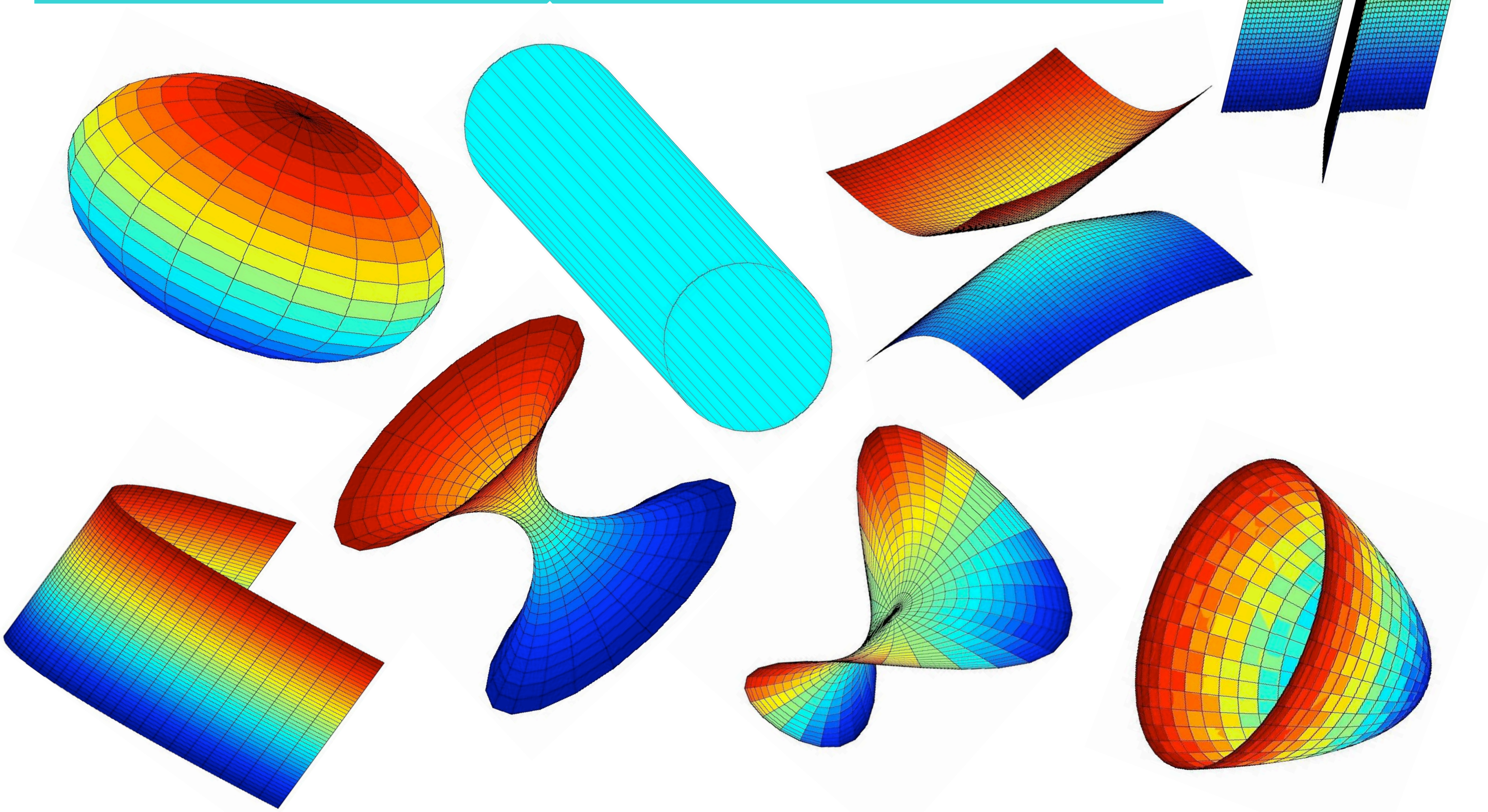
hyperbolic cylinder

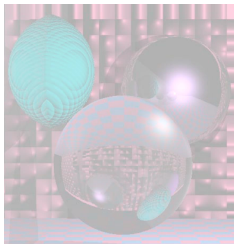


$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

General Quadrics in General Position







General Quadric Equation

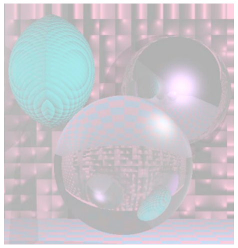
$$ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz + 2gx + 2hy + 2jz + k = 0$$

Ray Equations

$$x(t) = x_0 + tx_d \qquad x_d = x_1 - x_0$$
$$y(t) = y_0 + ty_d \qquad y_d = y_1 - y_0$$
$$z(t) = z_0 + tz_d \qquad z_d = z_1 - z_0$$

$$a(x_0 + tx_d)^2 + b(y_0 + ty_d)^2 + c(z_0 + tz_d)^2$$
$$+2d(x_0 + tx_d)(y_0 + ty_d) + 2e(y_0 + ty_d)(z_0 + tz_d) + 2f(x_0 + tx_d)(z_0 + tz_d)$$
$$+2g(x_0 + tx_d) + 2h(y_0 + ty_d) + 2j(z_0 + tz_d) + k = 0$$

$$A = ax_d^2 + by_d^2 + cz_d^2 + 2dx_d y_d + 2ey_d z_d + 2fx_d z_d$$
$$B = 2ax_0 x_d + 2by_0 y_d + 2cz_0 z_d$$
$$+ 2d(x_0 y_d + x_d y_0) + 2e(y_0 z_d + y_d z_0) + 2f(x_0 z_d + x_d z_0) + 2gx_d + 2hy_d + 2jz_d$$
$$C = ax_0^2 + by_0^2 + cz_0^2 + 2dx_0 y_0 + 2ey_0 z_0 + 2fx_0 z_0 + 2gx_0 + 2hy_0 + 2jz_0 + k$$



Ray Quadric Intersection Quadratic Coefficients

A = a*xd*xd + b*yd*yd + c*zd*zd

+ 2[d*xd*yd + e*yd*zd + f*xd*zd

B = 2*[a*x0*xd + b*y0*yd + c*z0*zd

+ d*(x0*yd + xd*y0) + e*(y0*zd + yd*z0) + f*(x0*zd + xd*z0)

+ g*xd + h*yd + j*zd]

C = a*x0*x0 + b*y0*y0 + c*z0*z0

+ 2*[d*x0*y0 + e*y0*z0 + f*x0*z0 + g*x0 + h*y0 + j*z0] + k

Quadric Normals

$$Q(x,y,z) = ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz + 2gx + 2hy + 2jz + k$$

$$\frac{\partial Q}{\partial x} = 2ax + 2dy + 2fz + 2g = 2(ax + dy + fz + g)$$

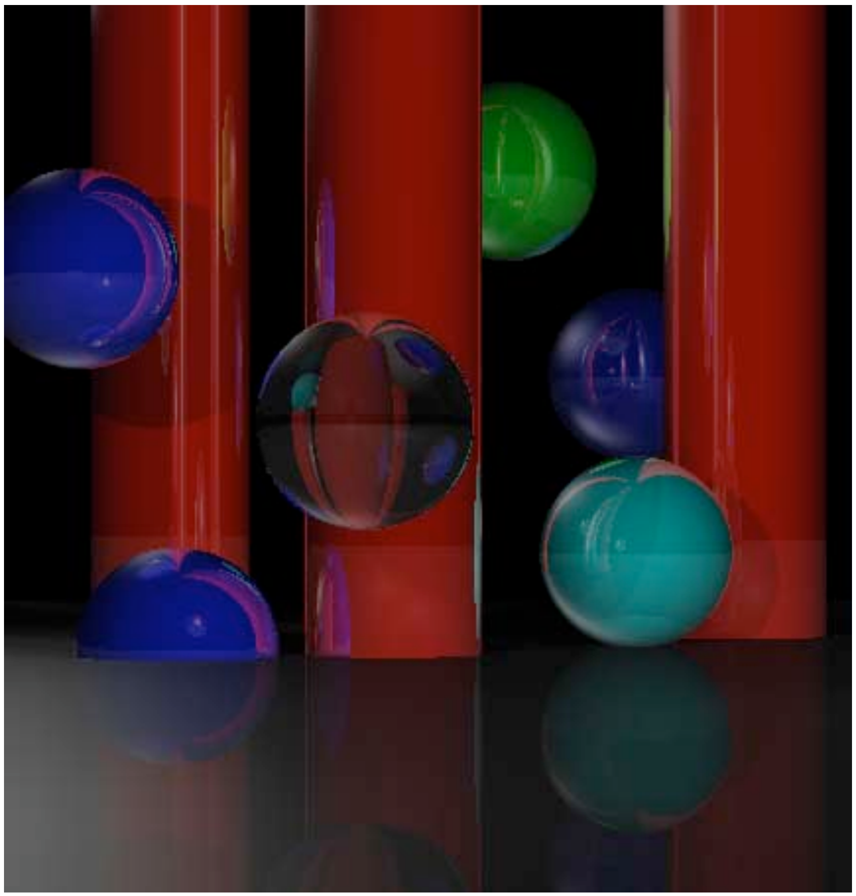
$$\frac{\partial Q}{\partial y} = 2by + 2dx + 2ez + 2h = 2(by + dx + ez + h)$$

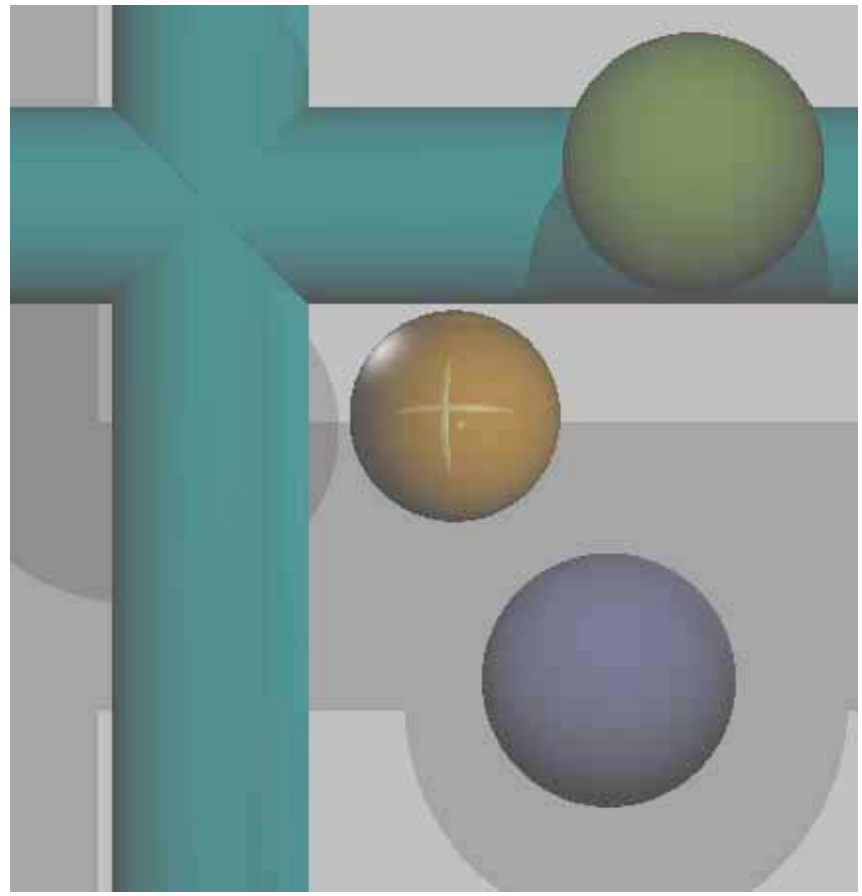
$$\frac{\partial Q}{\partial z} = 2cz + 2ey + 2fx + 2j = 2(cz + ey + fx + j)$$

$$N = \left(\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z} \right)$$

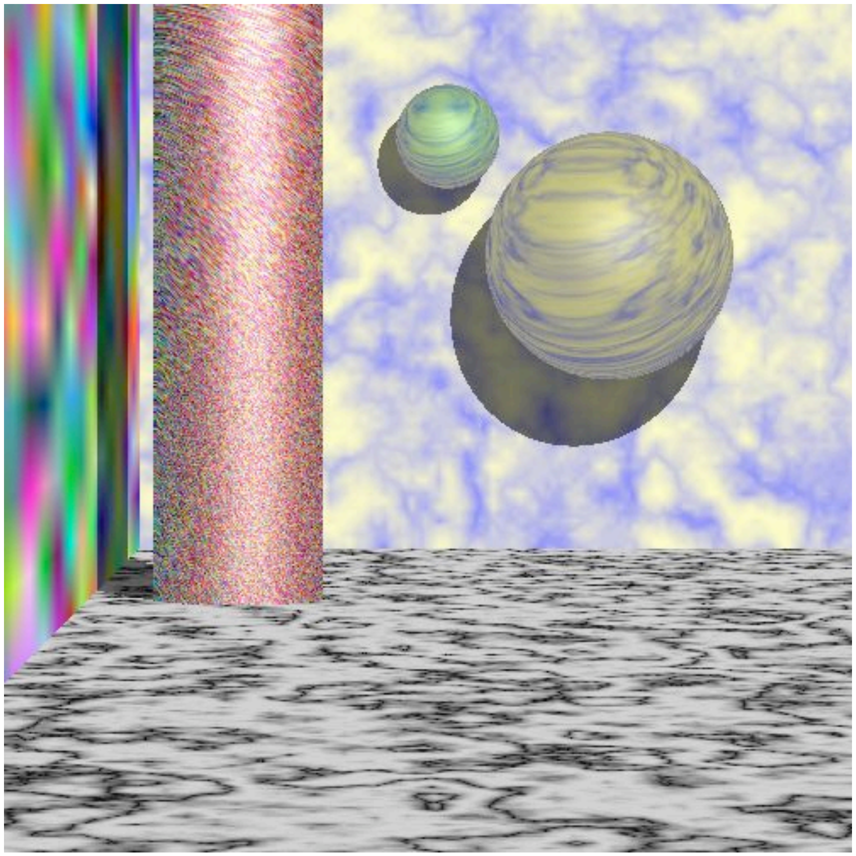
Normalize N and change its sign if necessary.

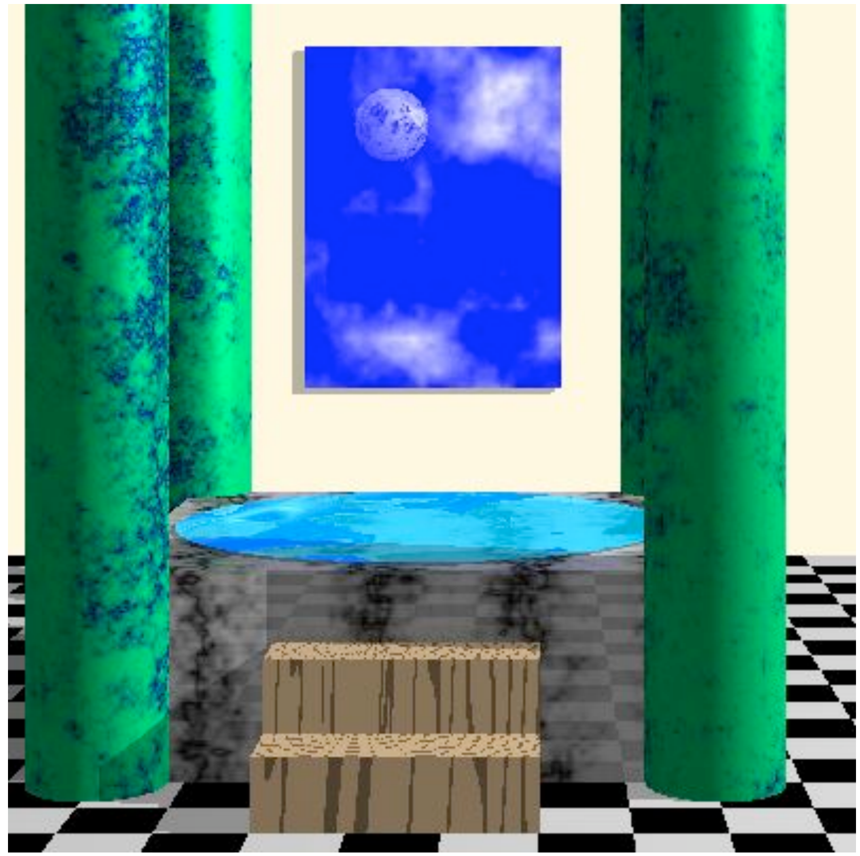
MyCylinders





Student Images





Student Images