CS 4300
Computer Graphics

Prof. Harriet Fell

Fall 2011

Lecture 11 – September 29, 2011

Today's Topics

- Linear Algebra Review
  - Matrices
  - Transformations

- New Linear Algebra
  - Homogeneous Coordinates

Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

- We use 2x2, 3x3, and 4x4 matrices in computer graphics.
- We'll start with a review of 2D matrices and transformations.
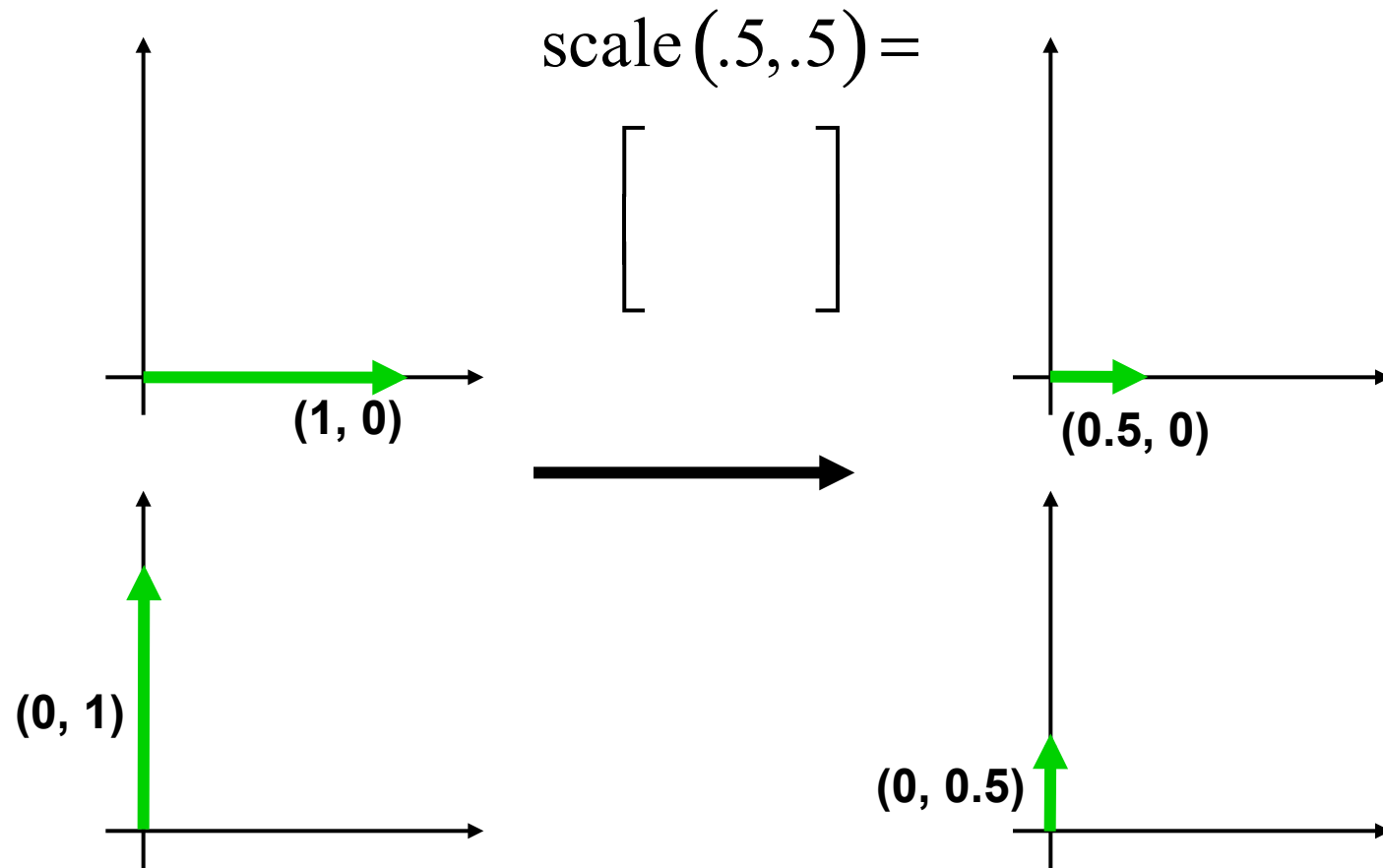
Basic 2D Linear Transforms

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$
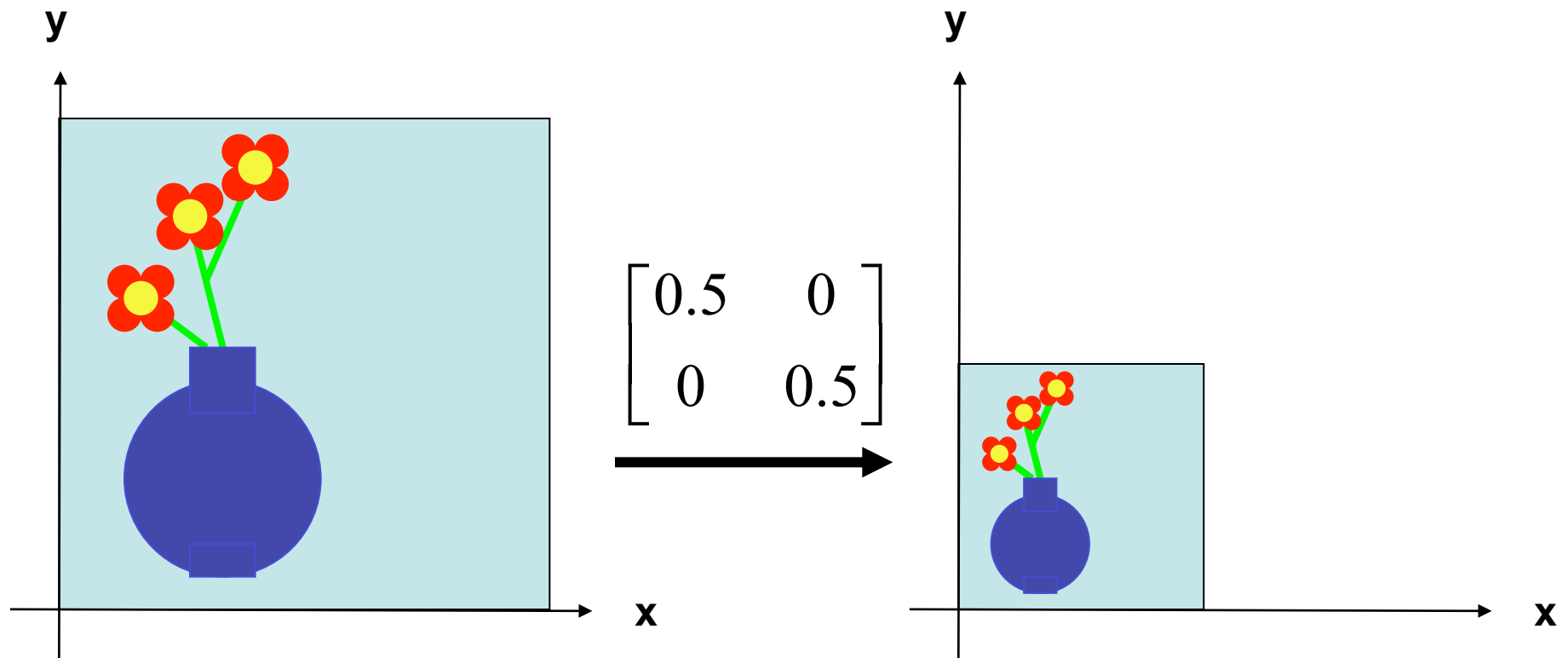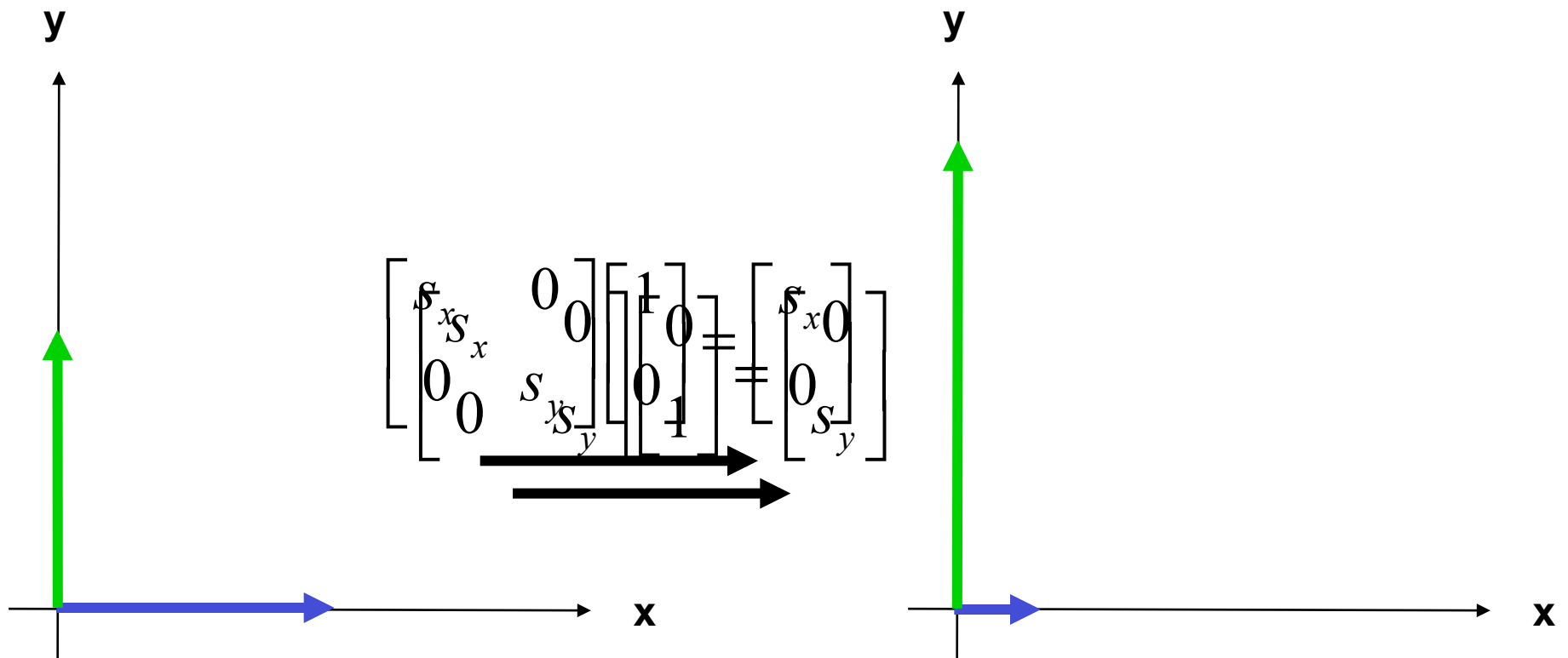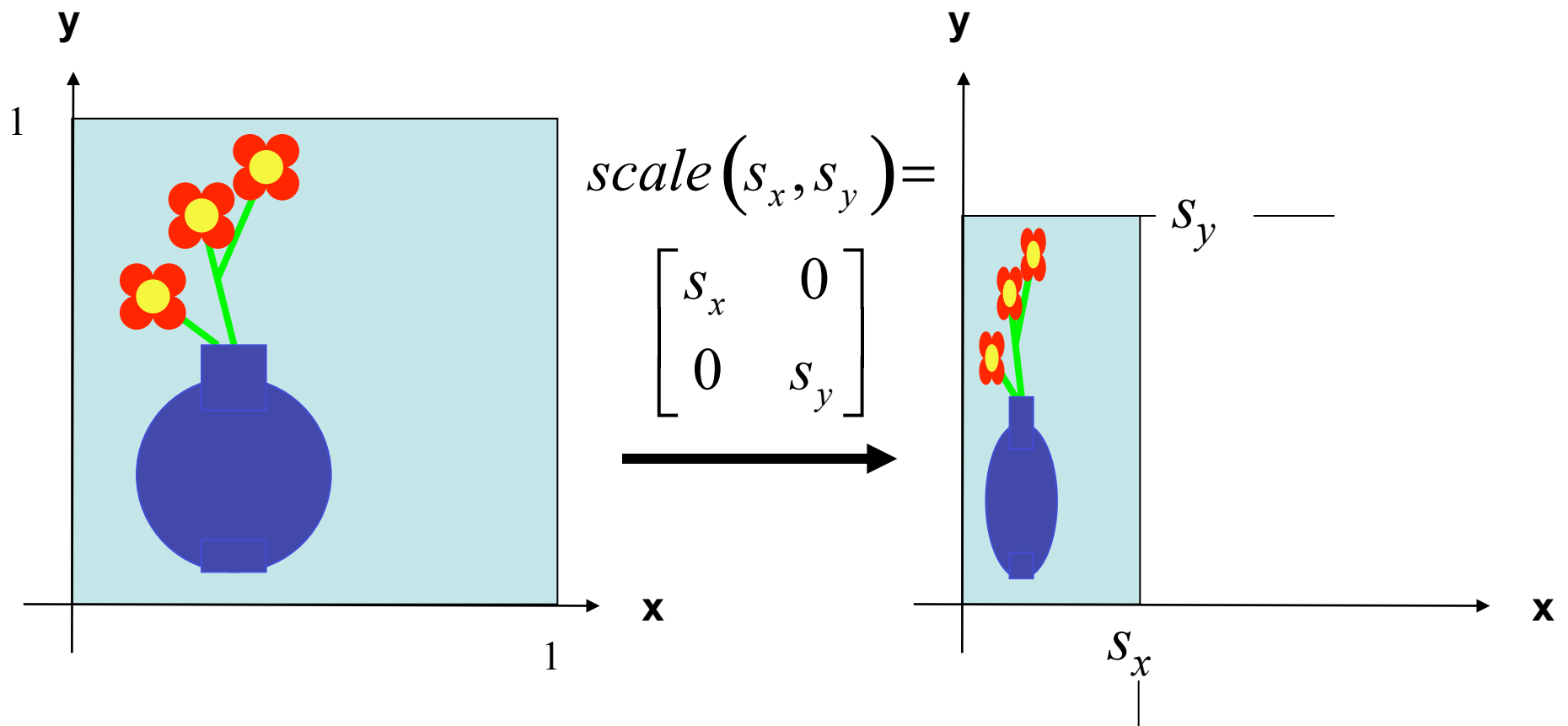
Scale by .5

$$\text{scale}(.5, .5) = \begin{bmatrix} & \\ & \end{bmatrix}$$
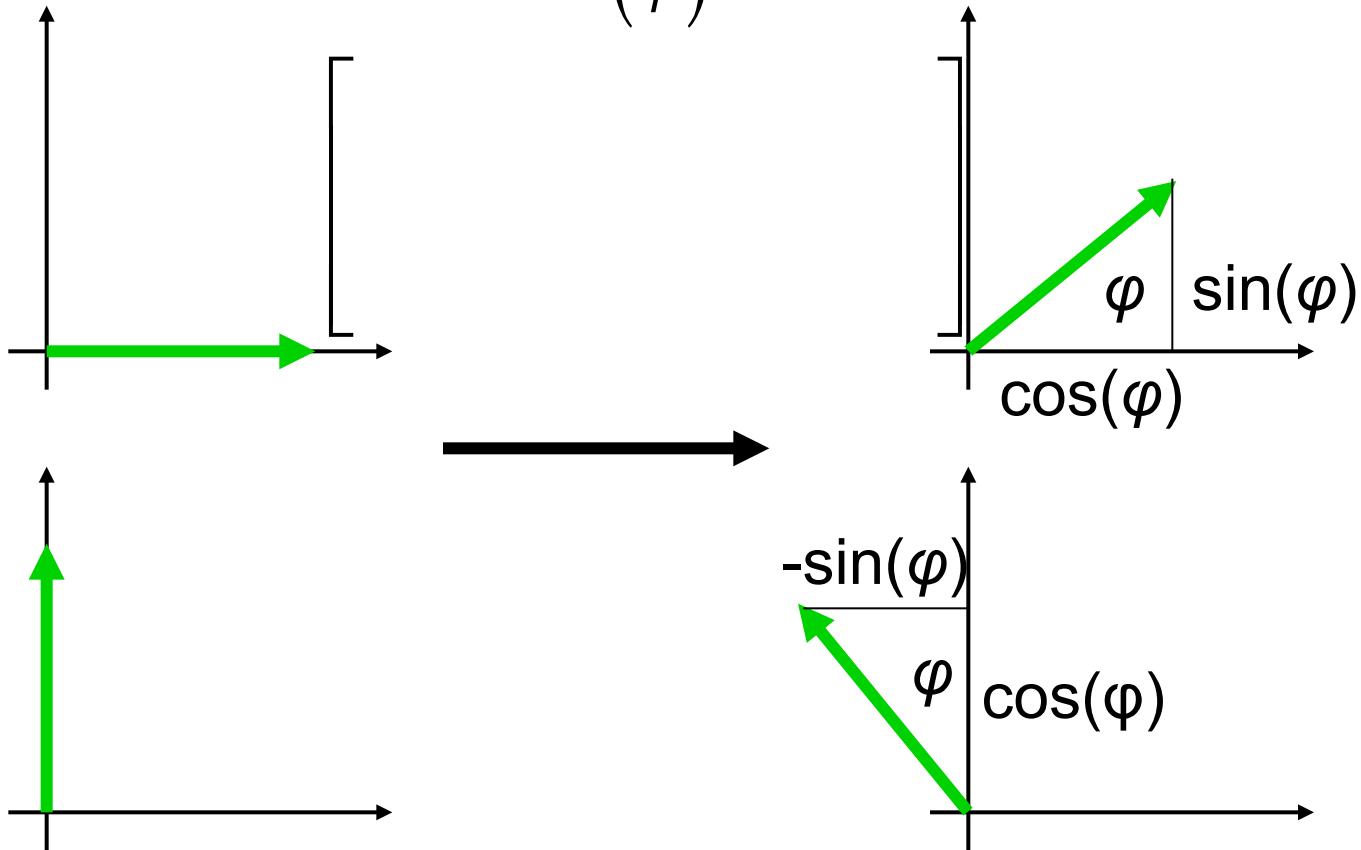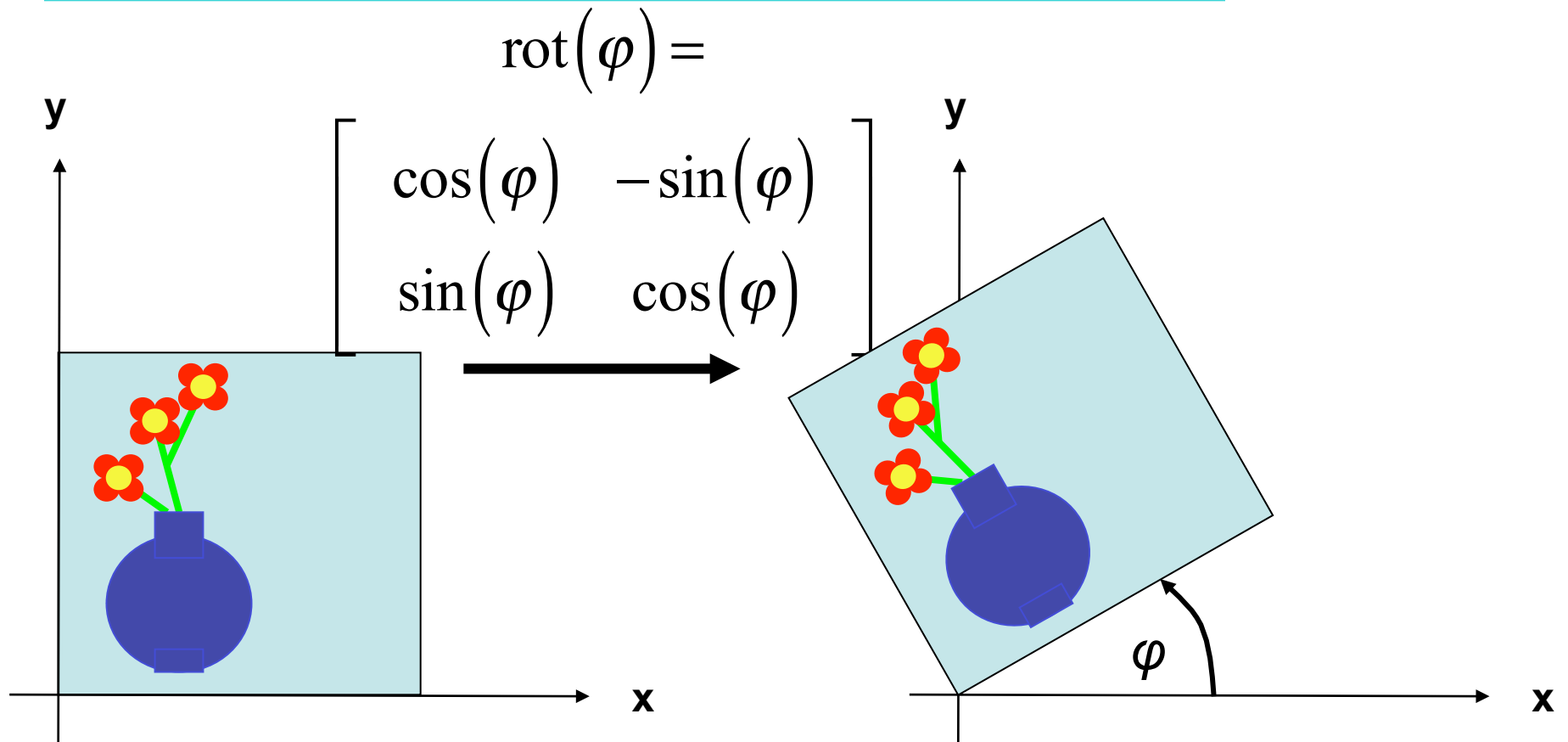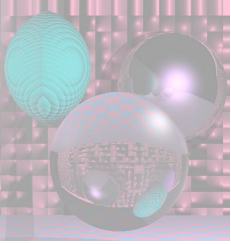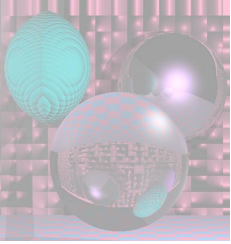
(1, 0)

(0.5, 0)

(0, 1)

(0, 0.5)

Scaling by .5

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

General Scaling

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

General Scaling

$$scale\left(s_x, s_y\right)=$$

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Rotation

$$\mathrm{rot}(\varphi) =$$

$$\varphi \quad \sin(\varphi)$$

$$\cos(\varphi)$$

$$-\sin(\varphi)$$

$$\varphi \quad \cos(\varphi)$$

Rotation

$$\text{rot}(\varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

Reflection in y-axis

reflect-y =

$$\begin{bmatrix} & \\ & \end{bmatrix}$$

Reflection in y-axis

$$\text{reflect-y} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection in x-axis

reflect-x $=$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection in x-axis
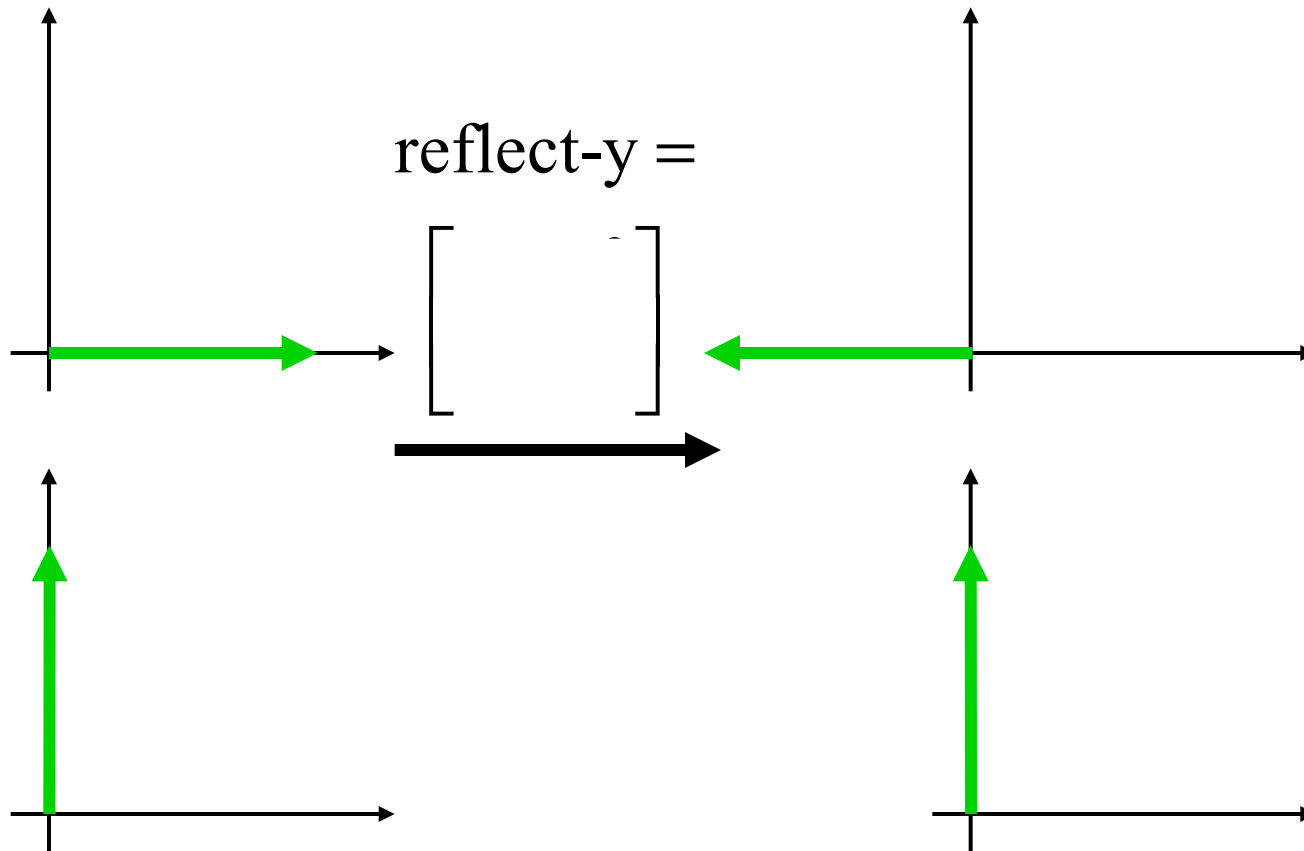
$$\text{reflect-x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
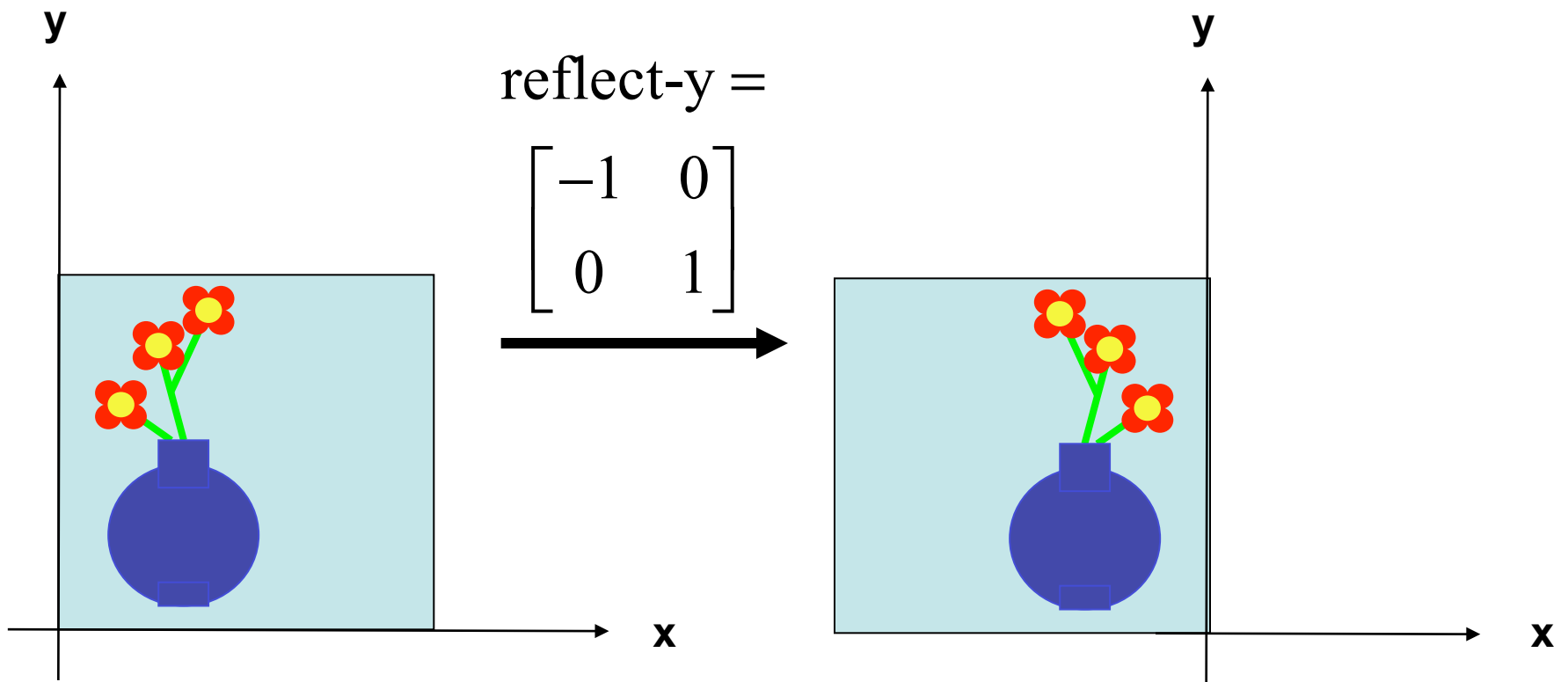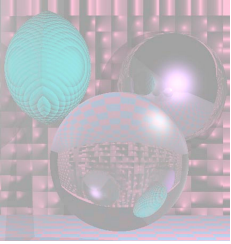
Shear-x
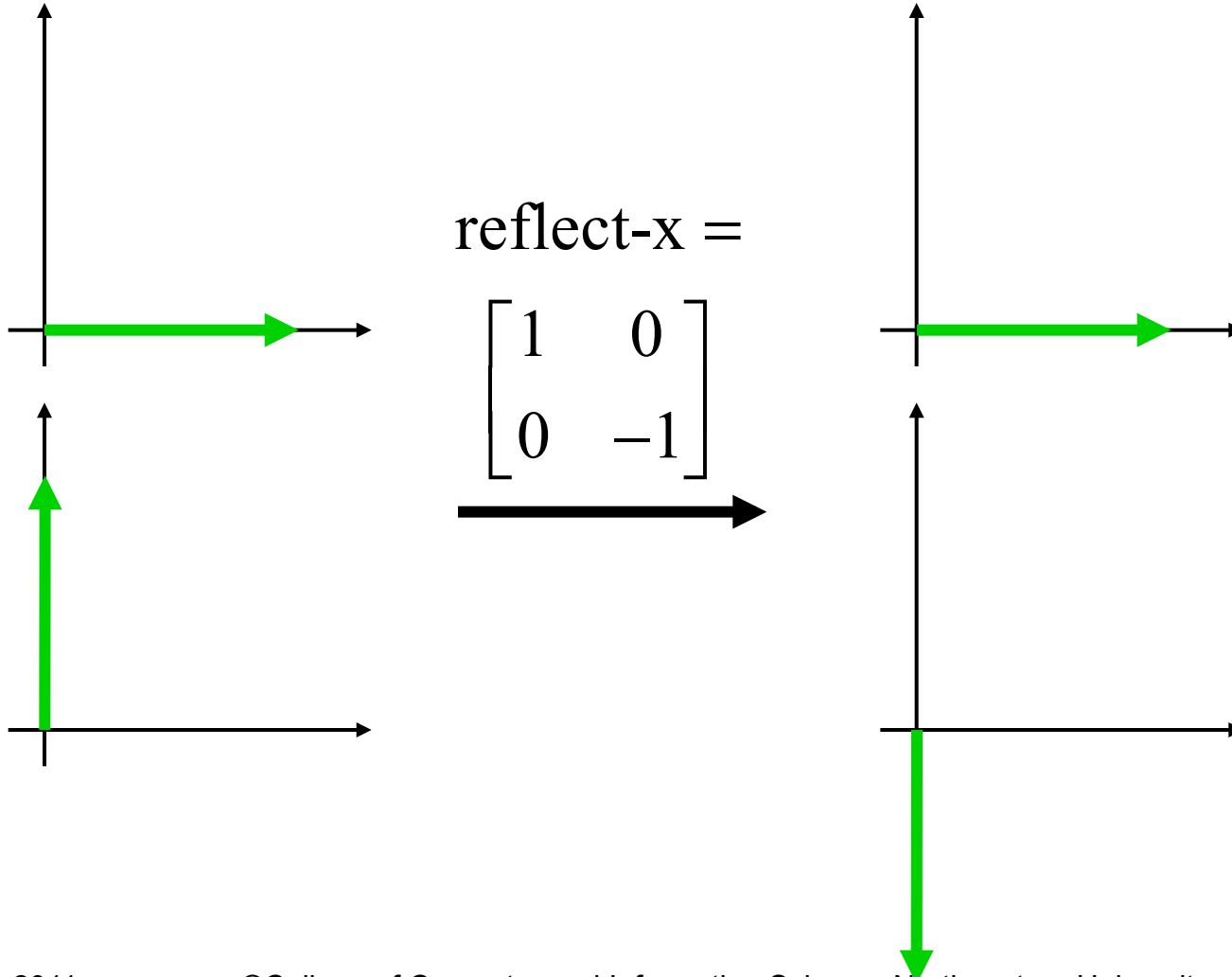
$$\text{shear-x}(s) = \begin{bmatrix} & \\ & \end{bmatrix}$$
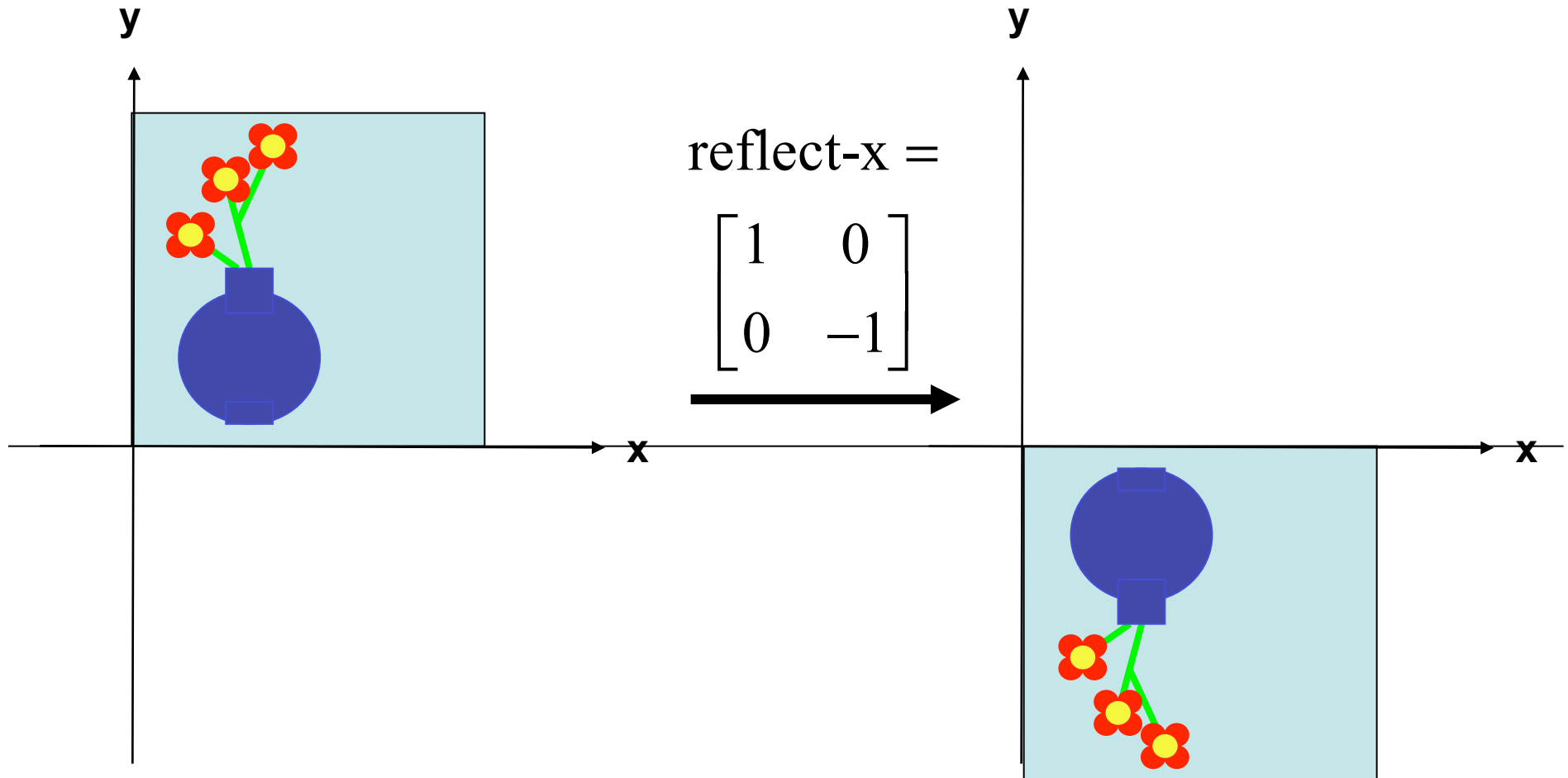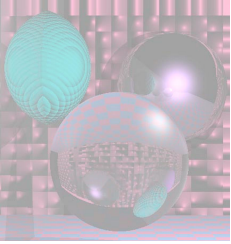
Shear x
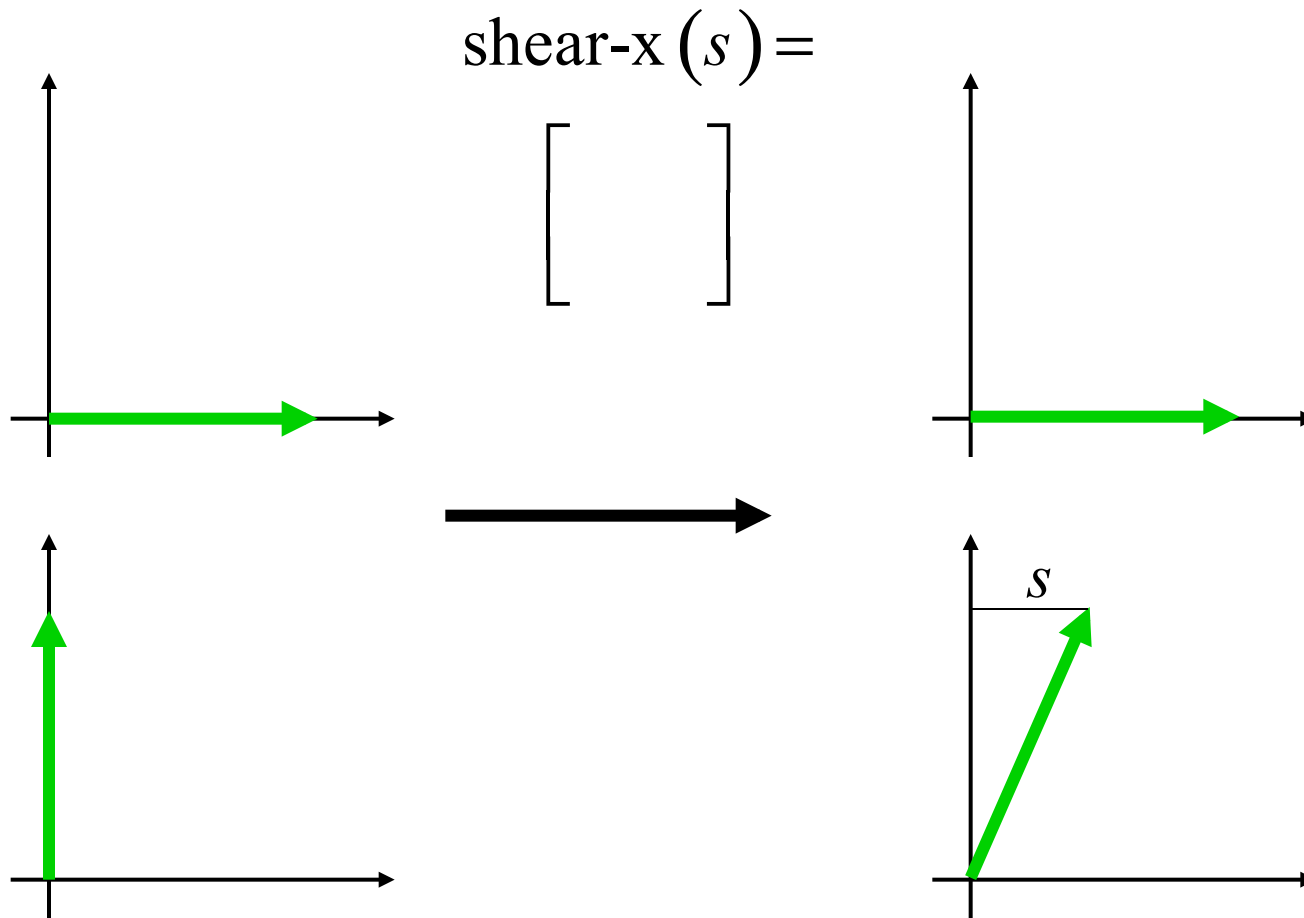
$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$
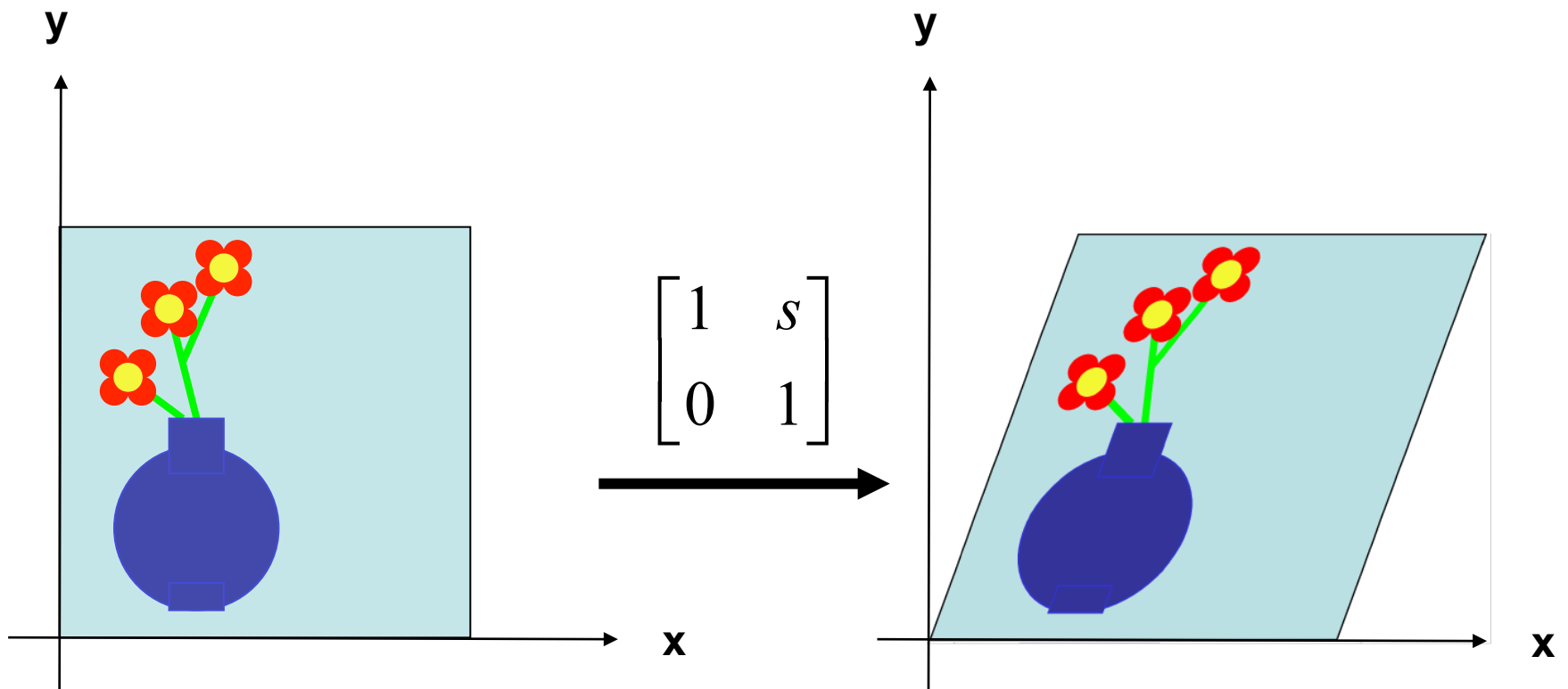
Shear-y

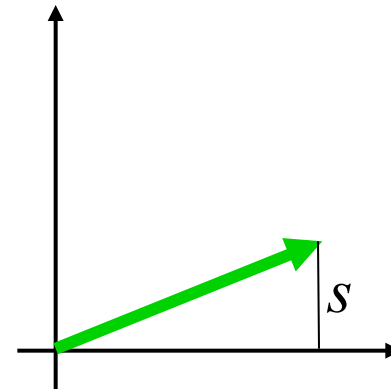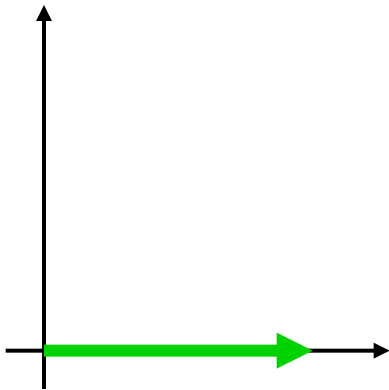$$\text{shear-y}(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

s

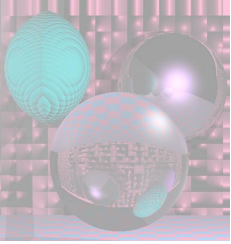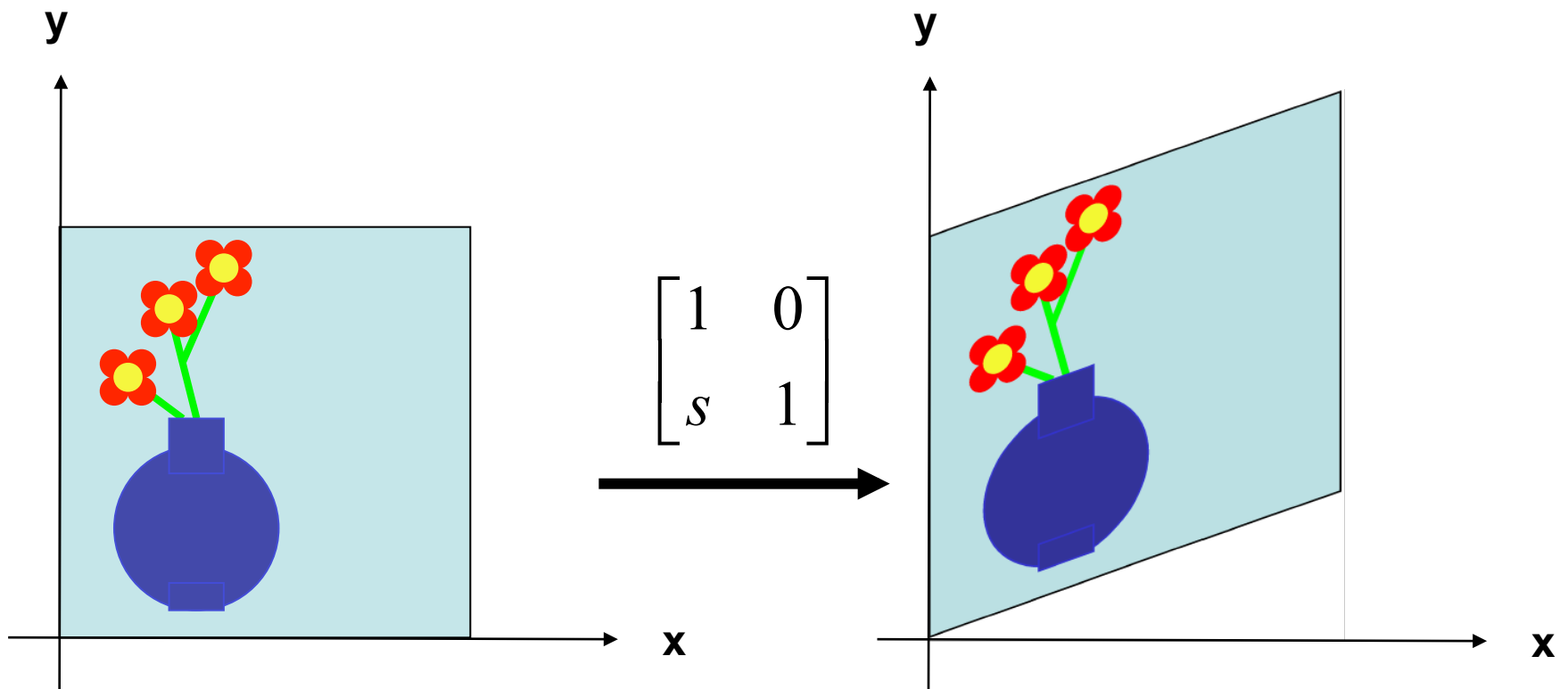# Shear y

$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$
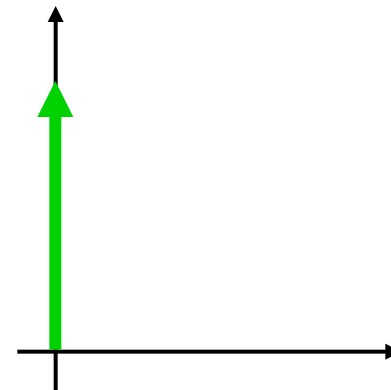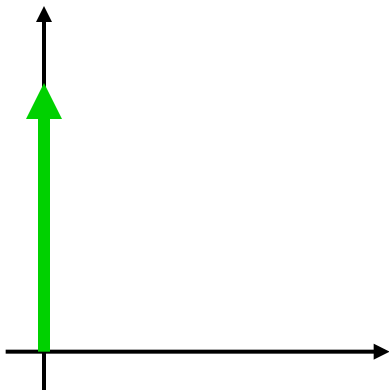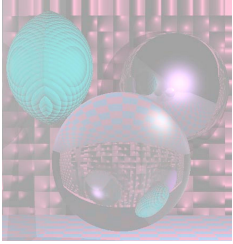
Linear Transformations

- Scale, Reflection, Rotation, and Shear are all linear transformations

- They satisfy: $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$
  - $\mathbf{u}$ and $\mathbf{v}$ are vectors
  - a and b are scalars

- If T is a linear transformation
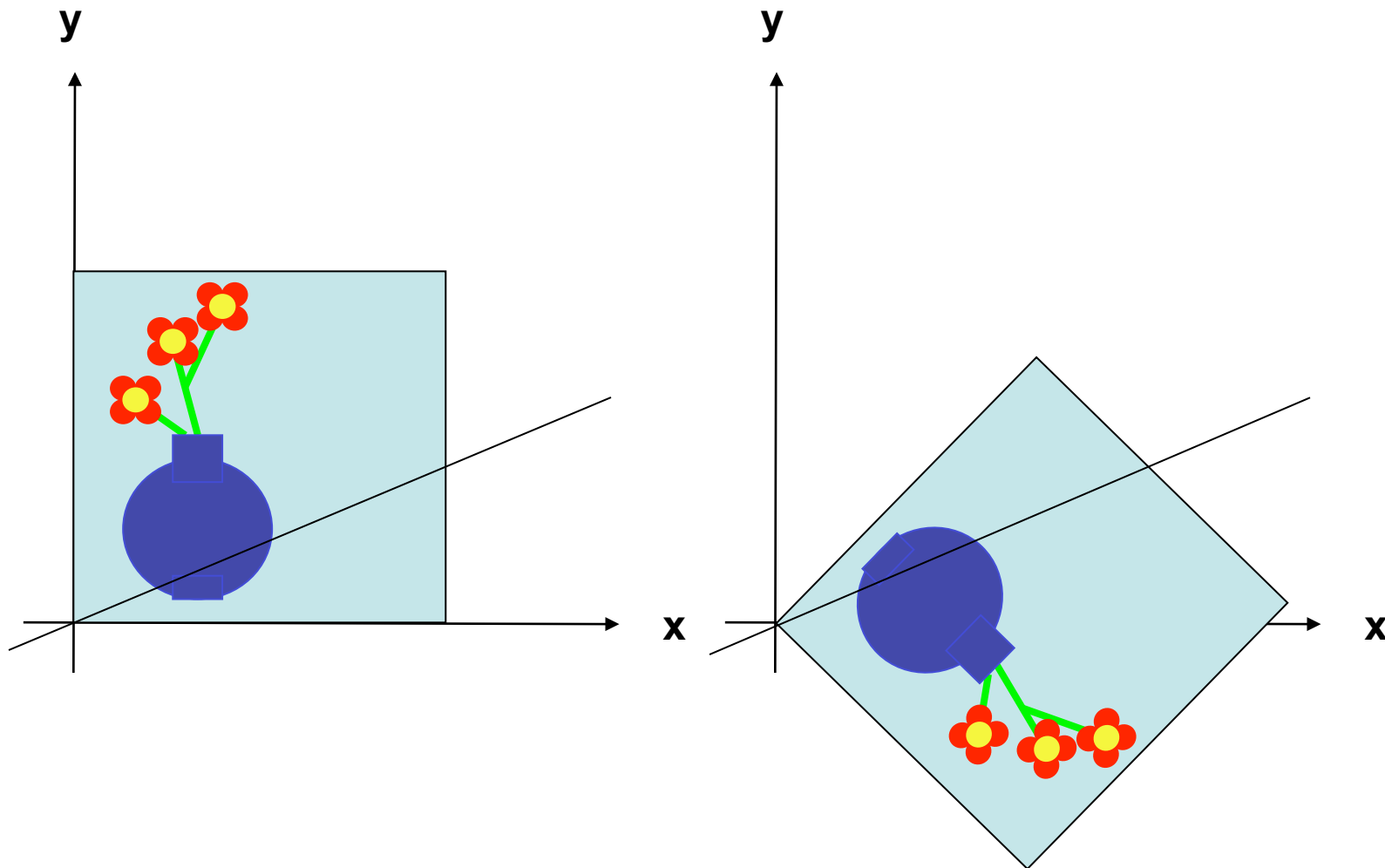 - $T((0, 0)) = (0, 0)$
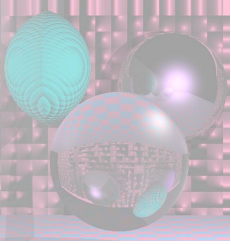
Composing Linear Transformations

- If T_1 and T_2 are transformations
  - $T_2 T_1(\mathbf{v}) =_{def} T_2(T_1(\mathbf{v}))$
- If T_1 and T_2 are linear and are represented by matrices M_1 and M_2
  - $T_2 T_1$ is represented by $M_2 M_1$
  - $T_2 T_1(\mathbf{v}) = T_2(T_1(\mathbf{v})) = (M_2 M_1)(\mathbf{v})$

Reflection About an Arbitrary Line (through the origin)

Reflection as a Composition

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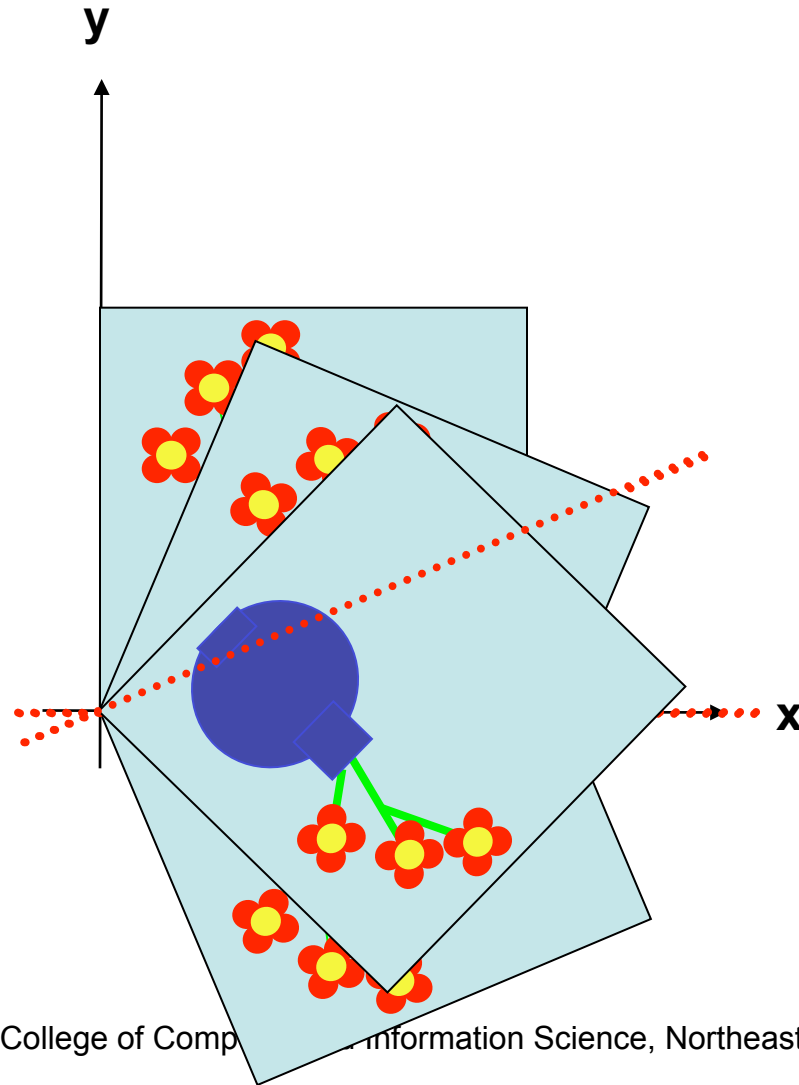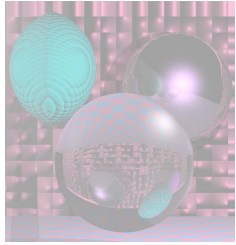
Decomposing
Linear Transformations

- Any 2D Linear Transformation can be decomposed into the product of a rotation, a scale, and a rotation if the scale can have negative numbers.

- $M = R_1 S R_2$

Rotation about an Arbitrary Point

y

y

φ

φ

x

x

This is not a linear transformation. The origin moves.

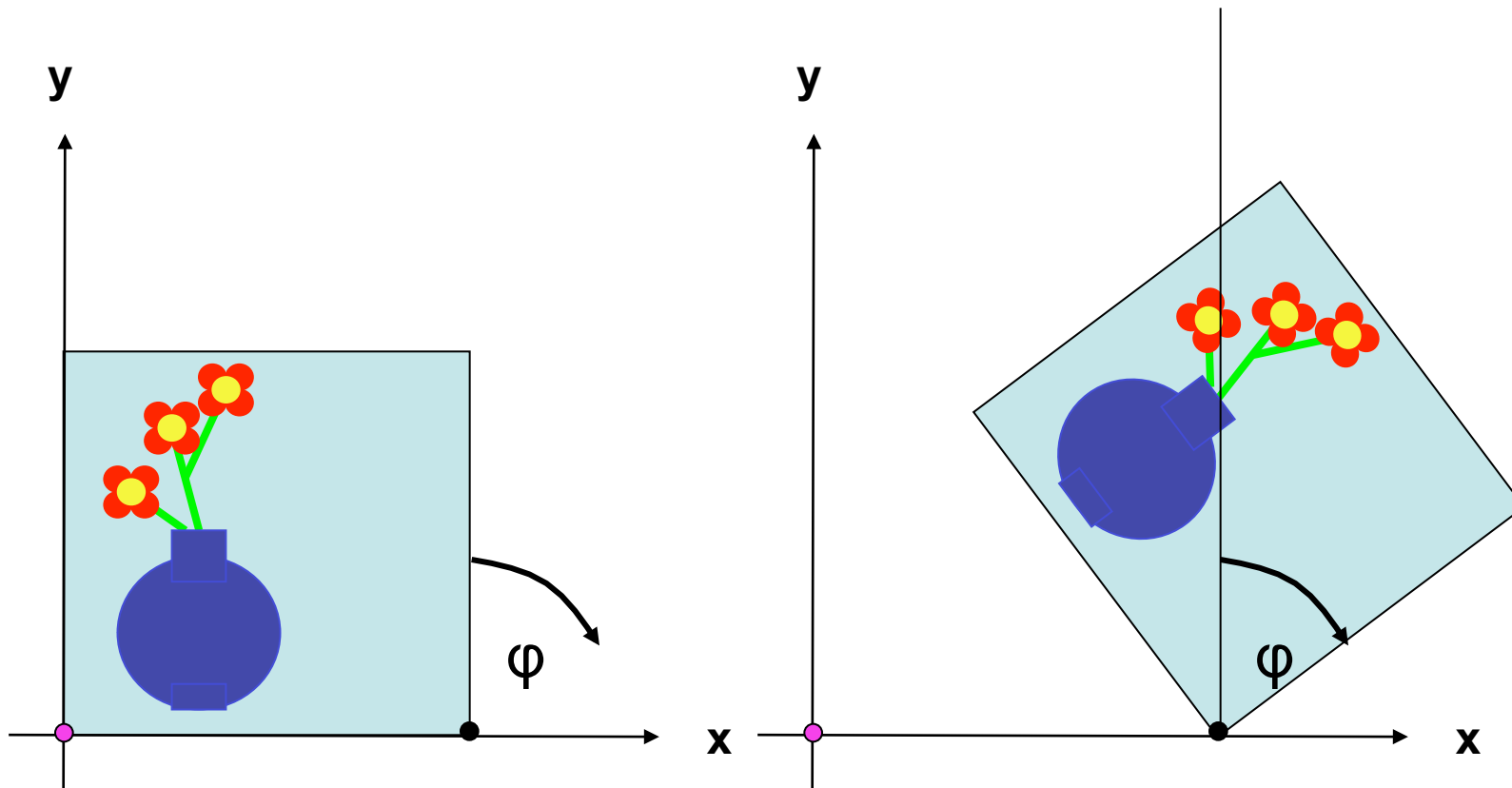# Translation

$$(x, y) \rightarrow (x+a, y+b)$$
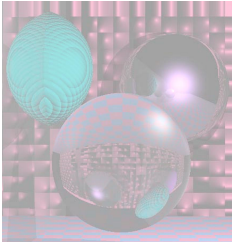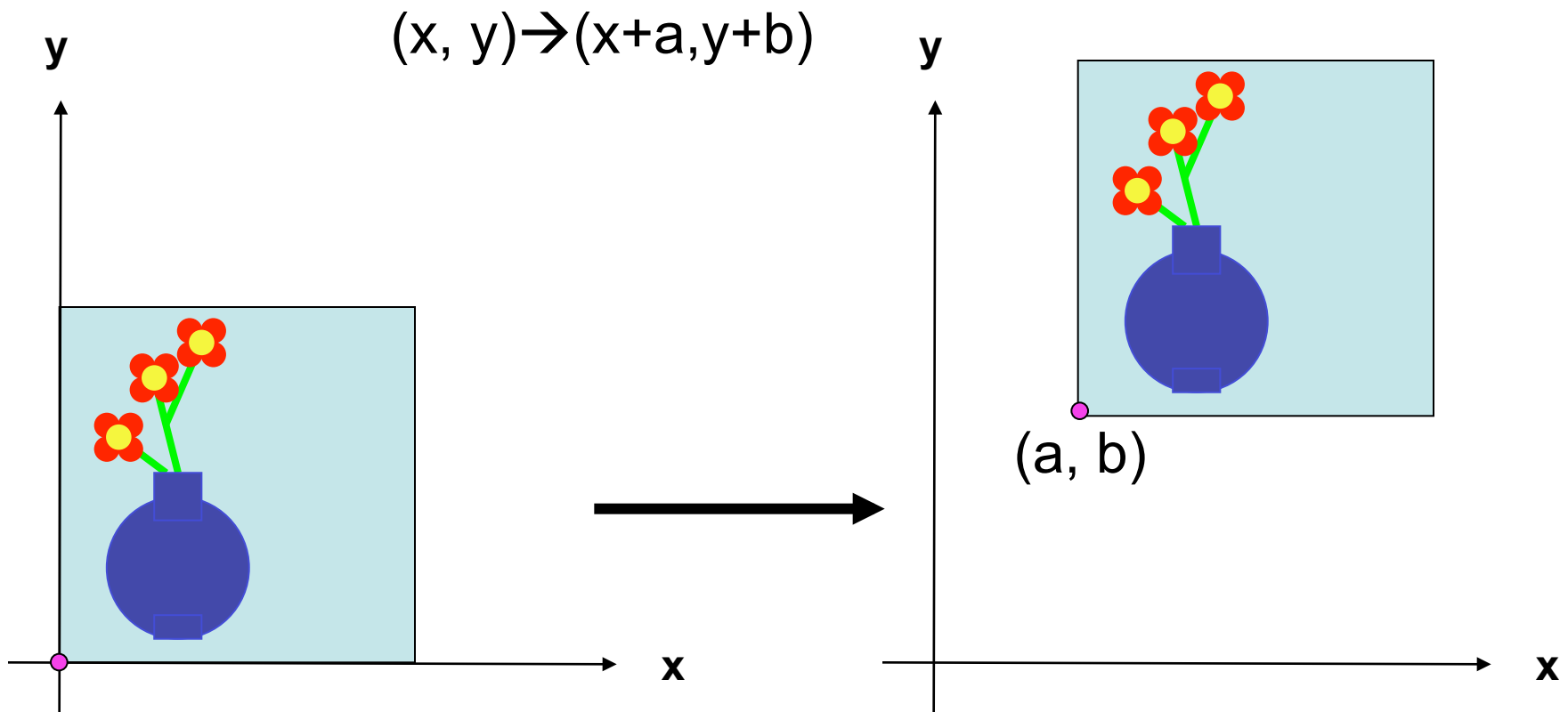
This is not a linear transformation. The origin moves.

Homogeneous Coordinates

Embed the xy-plane in R^3 at $z = 1$.

$(x, y) \rightarrow (x, y, 1)$

2D Linear Transformations as 3D Matrices

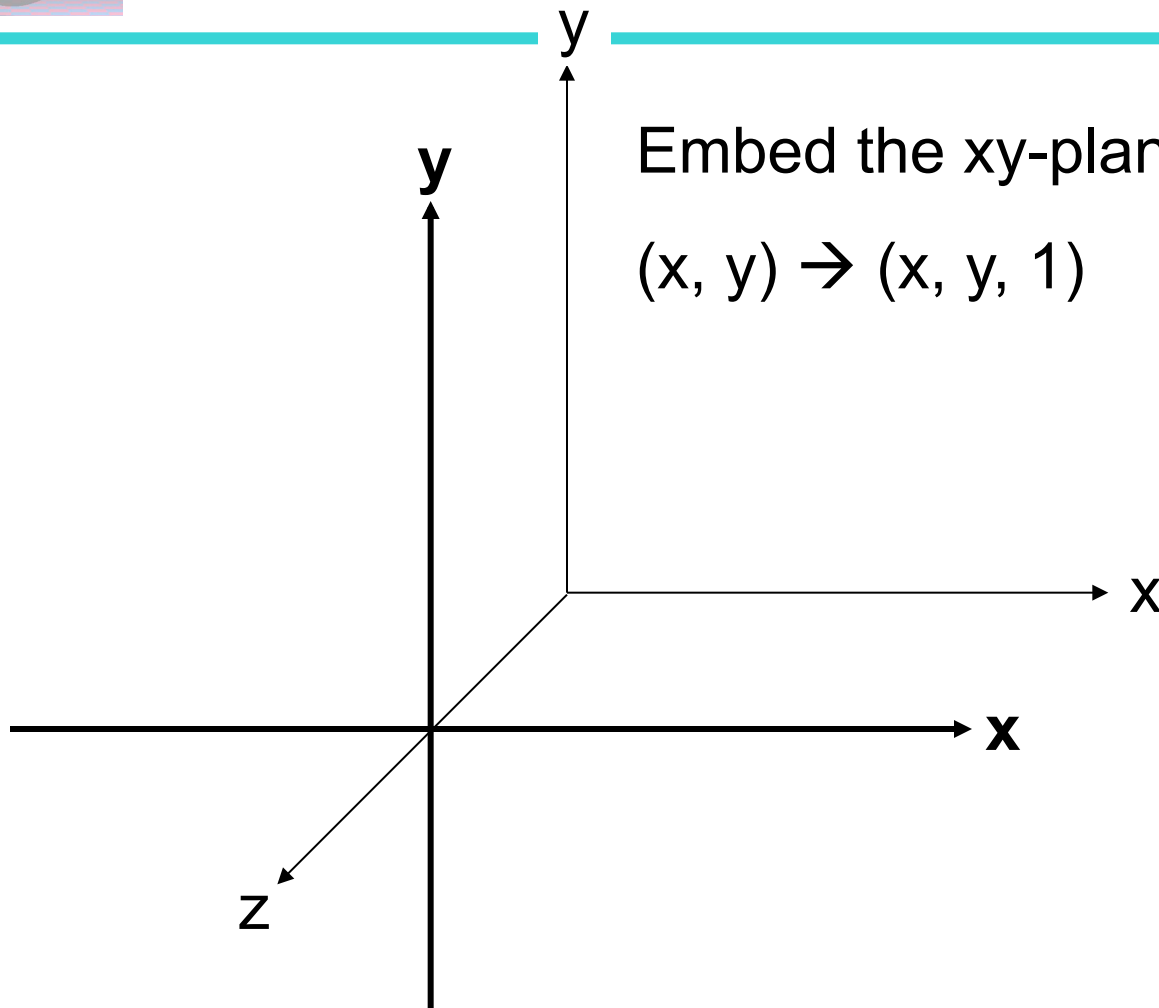Any 2D linear transformation can be represented by a 2x2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

or a 3x3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ 1 \end{bmatrix}$$
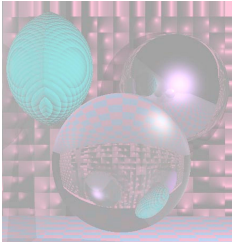
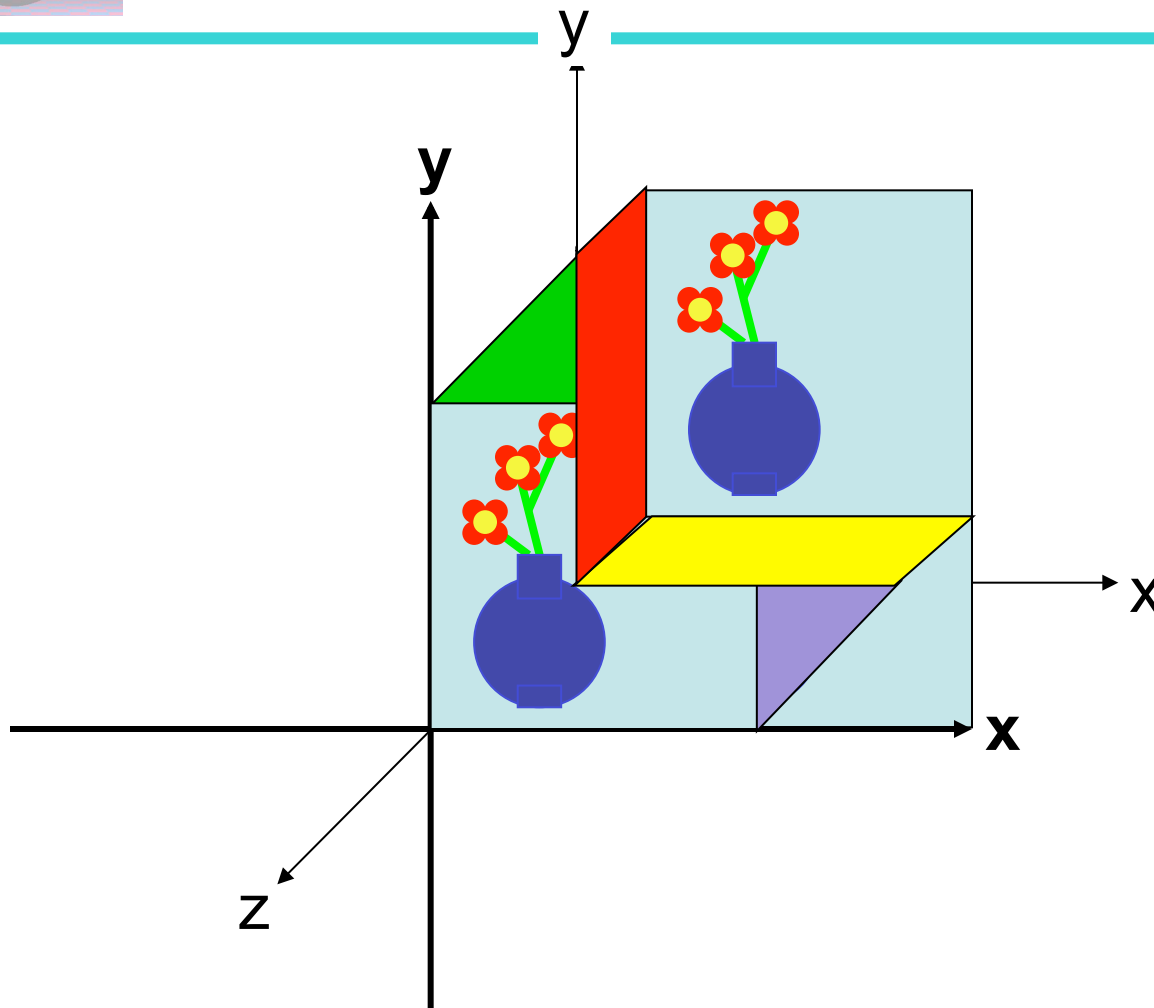# 2D Linear Translations as 3D Matrices

Any 2D translation can be represented by a 3x3 matrix.

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

This is a 3D shear that acts as a translation on the plane z = 1.

Translation as a Shear

2D Affine Transformations

- An *affine transformation* is any transformation that preserves co-linearity (i.e., all points lying on a line initially still lie on a line after transformation) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after transformation).

- With homogeneous coordinates, we can represent all 2D affine transformations as 3D linear transformations.

- We can then use matrix multiplication to transform objects.
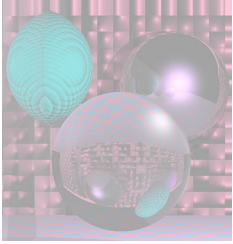
Rotation about
an Arbitrary Point

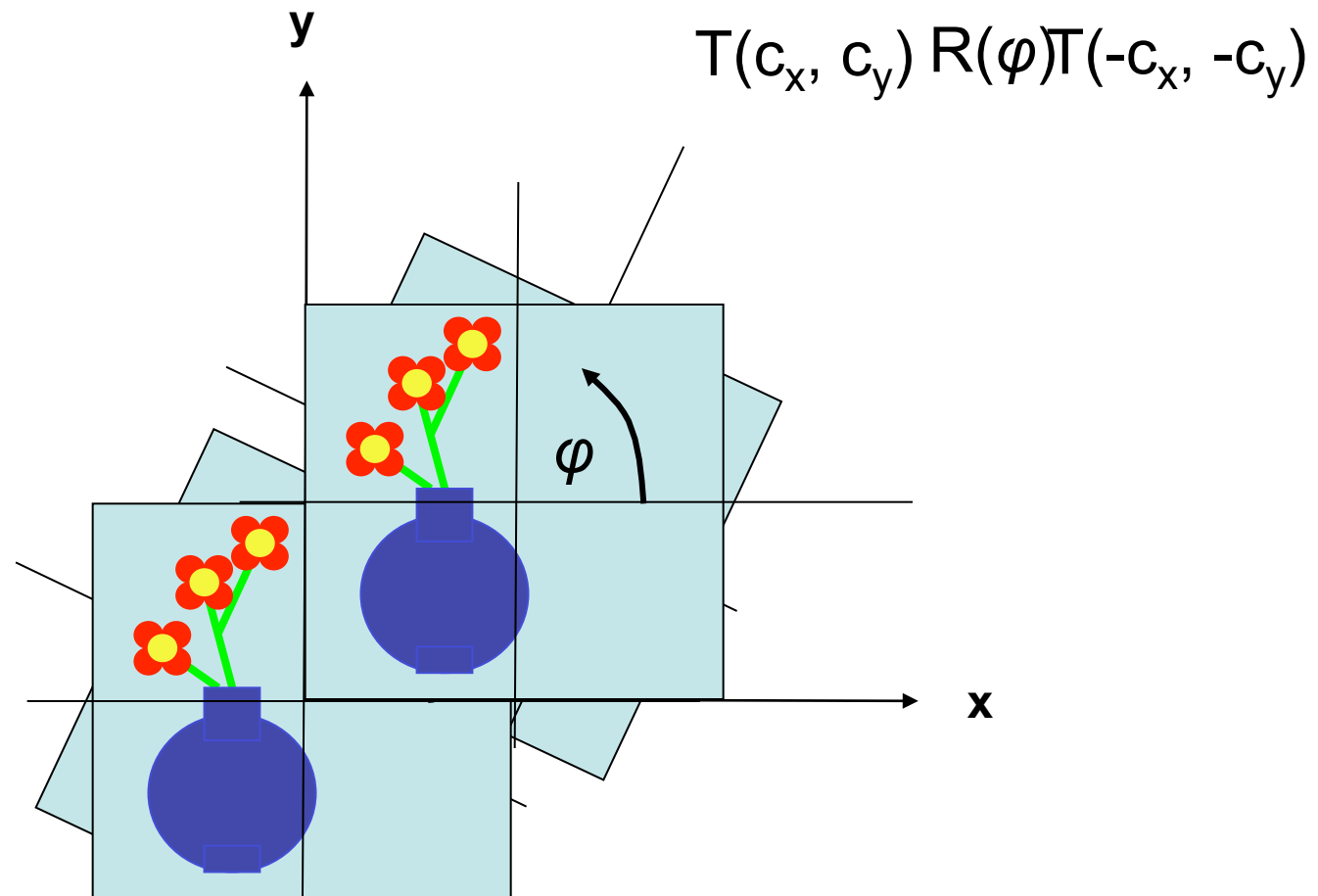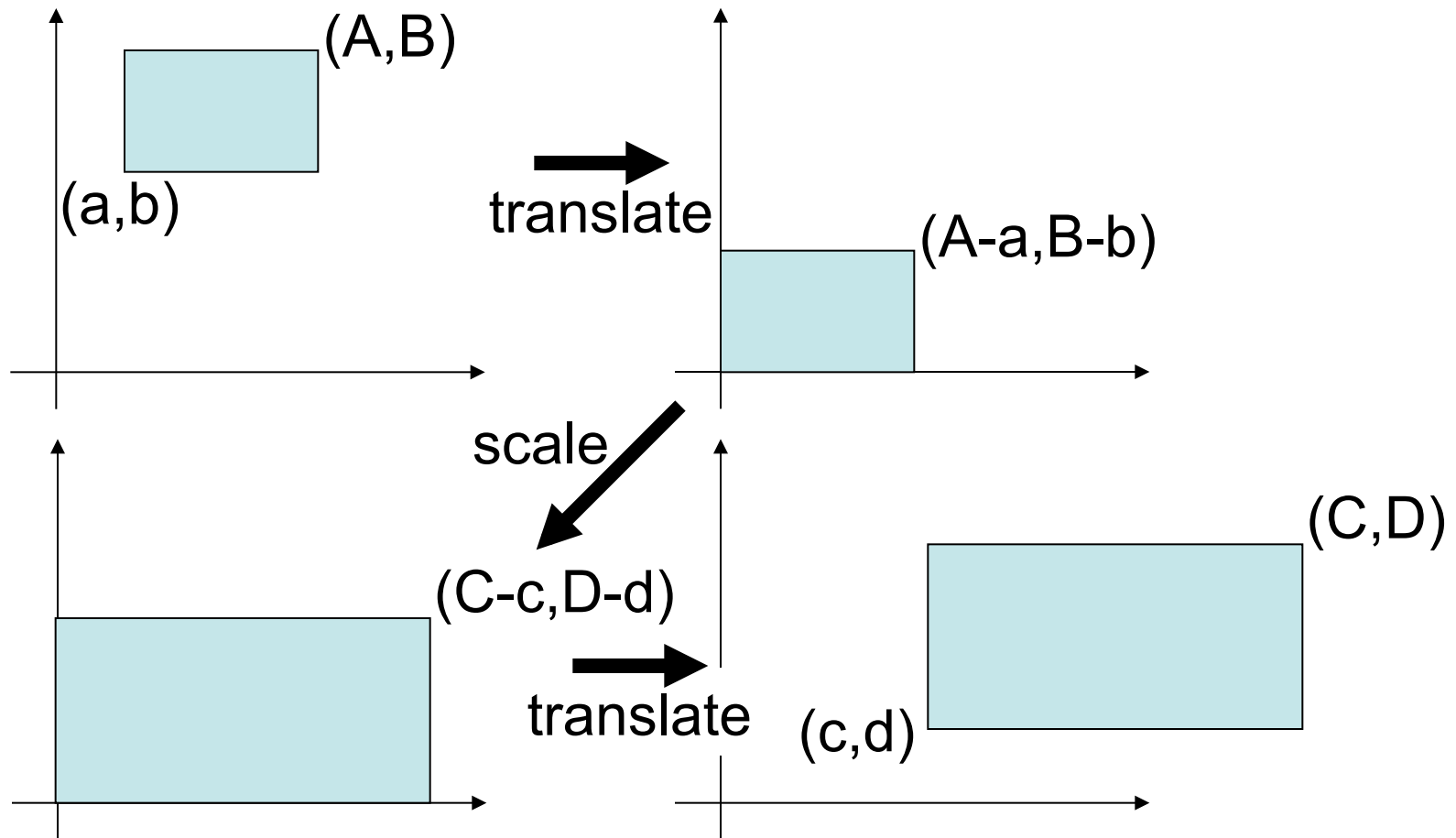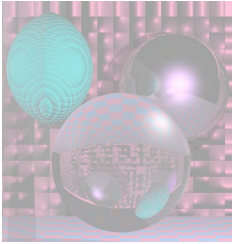# Rotation about an Arbitrary Point

$$T(c_x, c_y) R(\varphi) T(-c_x, -c_y)$$

Windowing Transforms

(A,B)

(a,b)

translate

(A-a,B-b)

scale

(C-c,D-d)

(C,D)

translate

(c,d)
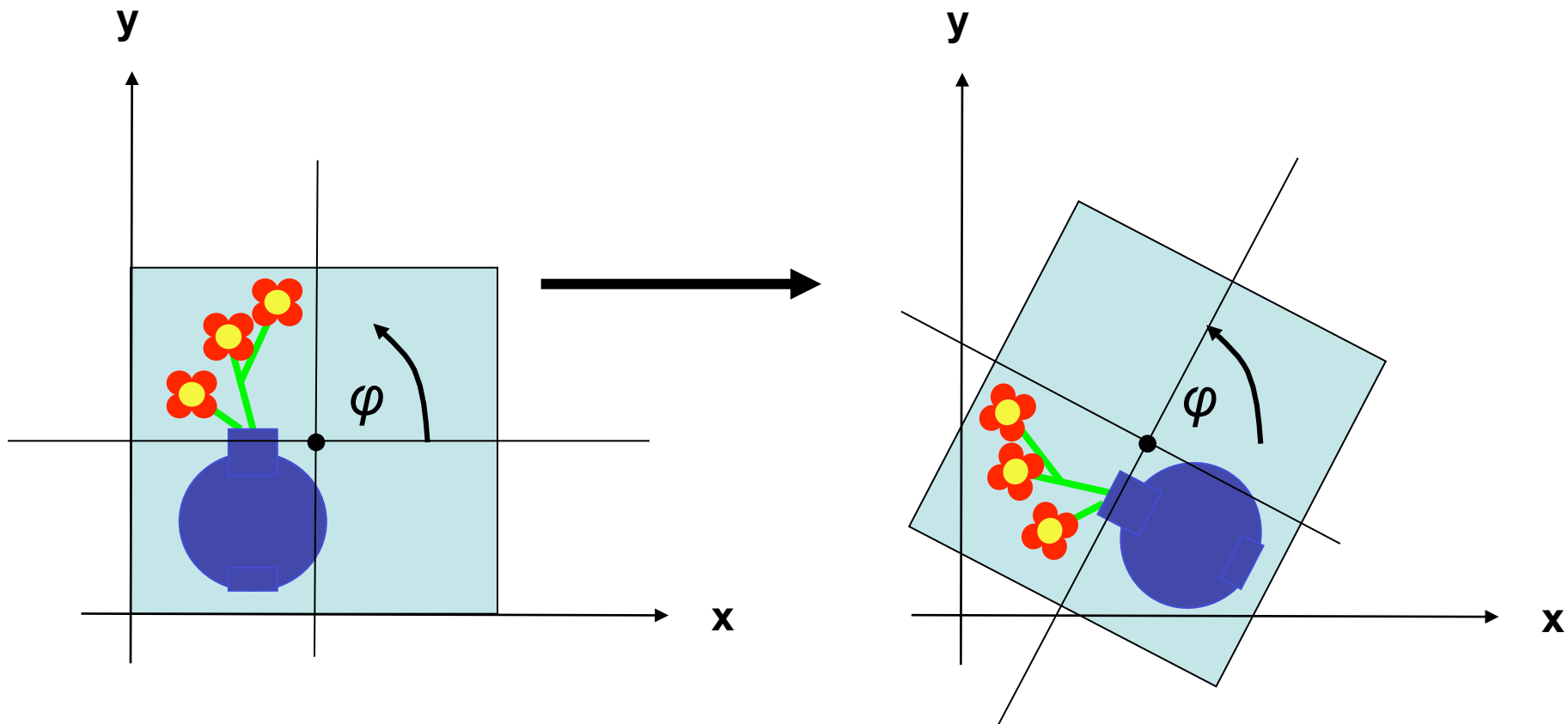
3D Transformations

Remember:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \leftrightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A 3D linear transformation can be represented by a 3x3 matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \leftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
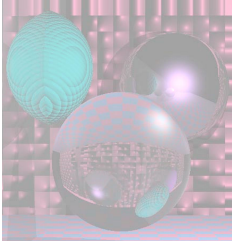
3D Affine Transformations

$$\text{scale}\left(s_x, s_y, s_z\right) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{translate}\left(t_x, t_y, t_z\right) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Rotations

$$\text{rotate}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate}_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$