3 Hypothetical and General Judgments

A *hypothetical judgment* expresses an entailment between one or more hypotheses and a conclusion. We will consider two notions of entailment, called *derivability* and *admissibility*. Both express a form of entailment, but they differ in that derivability is stable under extension with new rules, admissibility is not. A *general judgment* expresses the universality, or genericity, of a judgment. There are two forms of general judgment, the *generic* and the *parametric*. The generic judgment expresses generality with respect to all substitution instances for variables in a judgment. The parametric judgment expresses generality with respect to renamings of symbols.

3.1 Hypothetical Judgments

The hypothetical judgment codifies the rules for expressing the validity of a conclusion conditional on the validity of one or more hypotheses. There are two forms of hypothetical judgment that differ according to the sense in which the conclusion is conditional on the hypotheses. One is stable under extension with more rules, and the other is not.

3.1.1 Derivability

For a given set $\mathcal{R}$ of rules, we define the *derivability* judgment, written $J_1, \ldots, J_k \vdash_{\mathcal{R}} K$, where each J_i and K are basic judgments, to mean that we may derive K from the *expansion* $\mathcal{R} \cup \{ J_1, \ldots, J_k \}$ of the rules $\mathcal{R}$ with the axioms

$$\frac{}{J_1} \qquad \cdots \qquad \frac{}{J_k}.$$

We treat the *hypotheses*, or *antecedents*, of the judgment, $J_1, \ldots, J_k$ as "temporary axioms," and derive the *conclusion*, or *consequent*, by composing rules in $\mathcal{R}$. Thus, evidence for a hypothetical judgment consists of a derivation of the conclusion from the hypotheses using the rules in $\mathcal{R}$.

We use capital Greek letters, usually Γ or Δ, to stand for a finite set of basic judgments, and write $\mathcal{R} \cup \Gamma$ for the expansion of $\mathcal{R}$ with an axiom corresponding to each judgment in Γ. The judgment $\Gamma \vdash_{\mathcal{R}} K$ means that K is derivable from rules $\mathcal{R} \cup \Gamma$, and the judgment $\vdash_{\mathcal{R}} \Gamma$ means that $\vdash_{\mathcal{R}} J$ for each J in Γ. An equivalent way of defining $J_1, \ldots, J_n \vdash_{\mathcal{R}} J$ is

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to say that the rule

$$\frac{J_1 \quad \dots \quad J_n}{J} \tag{3.1}$$

is *derivable* from $\mathcal{R}$, which means that there is a derivation of J composed of the rules in $\mathcal{R}$ augmented by treating $J_1, \dots, J_n$ as axioms.

For example, consider the derivability judgment

$$a \ \mathsf{nat} \vdash_{(2.2)} \mathtt{succ}(\mathtt{succ}(a)) \ \mathsf{nat} \tag{3.2}$$

relative to rules (2.2). This judgment is valid for *any* choice of object a, as shown by the derivation

$$\frac{\dfrac{a \ \mathsf{nat}}{\mathtt{succ}(a) \ \mathsf{nat}}}{\mathtt{succ}(\mathtt{succ}(a)) \ \mathsf{nat}} \tag{3.3}$$

which composes rules (2.2), starting with $a\ \mathsf{nat}$ as an axiom, and ending with $\mathtt{succ}(\mathtt{succ}(a))\ \mathsf{nat}$. Equivalently, the validity of (3.2) may also be expressed by stating that the rule

$$\frac{a \ \mathsf{nat}}{\mathtt{succ}(\mathtt{succ}(a)) \ \mathsf{nat}} \tag{3.4}$$

is derivable from rules (2.2).

It follows directly from the definition of derivability that it is stable under extension with new rules.

Theorem 3.1 (Stability)**.** *If* $\Gamma \vdash_{\mathcal{R}} J$*, then* $\Gamma \vdash_{\mathcal{R} \cup \mathcal{R}'} J$.

Proof Any derivation of J from $\mathcal{R} \cup \Gamma$ is also a derivation from $(\mathcal{R} \cup \mathcal{R}') \cup \Gamma$, because any rule in $\mathcal{R}$ is also a rule in $\mathcal{R} \cup \mathcal{R}'$. □

Derivability enjoys a number of *structural properties* that follow from its definition, independently of the rules $\mathcal{R}$ in question.

Reflexivity Every judgment is a consequence of itself: $\Gamma, J \vdash_{\mathcal{R}} J$. Each hypothesis justifies itself as conclusion.

Weakening If $\Gamma \vdash_{\mathcal{R}} J$, then $\Gamma, K \vdash_{\mathcal{R}} J$. Entailment is not influenced by un-exercised options.

Transitivity If $\Gamma, K \vdash_{\mathcal{R}} J$ and $\Gamma \vdash_{\mathcal{R}} K$, then $\Gamma \vdash_{\mathcal{R}} J$. If we replace an axiom by a derivation of it, the result is a derivation of its consequent without that hypothesis.

Reflexivity follows directly from the meaning of derivability. Weakening follows directly from the definition of derivability. Transitivity is proved by rule induction on the first premise.

3.1.2 Admissibility

Admissibility, written $\Gamma \models_{\mathcal{R}} J$, is a weaker form of hypothetical judgment stating that $\vdash_{\mathcal{R}} \Gamma$ implies $\vdash_{\mathcal{R}} J$. That is, the conclusion J is derivable from rules $\mathcal{R}$ when the assumptions Γ are all derivable from rules $\mathcal{R}$. In particular if any of the hypotheses are *not* derivable relative to $\mathcal{R}$, then the judgment is *vacuously* true. An equivalent way to define the judgment $J_1, \ldots, J_n \models_{\mathcal{R}} J$ is to state that the rule

$$\frac{J_1 \quad \ldots \quad J_n}{J} \tag{3.5}$$

is *admissible* relative to the rules in $\mathcal{R}$. Given any derivations of $J_1, \ldots, J_n$ using the rules in $\mathcal{R}$, we may build a derivation of J using the rules in $\mathcal{R}$.

For example, the admissibility judgment

$$\texttt{succ}(a)\ \textsf{even} \models_{(2.8)} a\ \textsf{odd} \tag{3.6}$$

is valid, because any derivation of $\texttt{succ}(a)$ even from rules (2.2) must contain a subderivation of a odd from the same rules, which justifies the conclusion. This fact can be proved by induction on rules (2.8). That judgment (3.6) is valid may also be expressed by saying that the rule

$$\frac{\texttt{succ}(a)\ \textsf{even}}{a\ \textsf{odd}} \tag{3.7}$$

is *admissible* relative to rules (2.8).

In contrast to derivability the admissibility judgment is *not* stable under extension to the rules. For example, if we enrich rules (2.8) with the axiom

$$\frac{}{\texttt{succ}(\texttt{zero})\ \textsf{even}}\ , \tag{3.8}$$

then rule (3.6) is *inadmissible*, because there is no composition of rules deriving $\texttt{zero}$ odd. Admissibility is as sensitive to which rules are *absent* from an inductive definition as it is to which rules are *present* in it.

The structural properties of derivability ensure that derivability is stronger than admissibility.

Theorem 3.2. *If* $\Gamma \vdash_{\mathcal{R}} J$, *then* $\Gamma \models_{\mathcal{R}} J$.

Proof Repeated application of the transitivity of derivability shows that if $\Gamma \vdash_{\mathcal{R}} J$ and $\vdash_{\mathcal{R}} \Gamma$, then $\vdash_{\mathcal{R}} J$. □

To see that the converse fails, note that

$$\texttt{succ}(\texttt{zero})\ \textsf{even} \nvdash_{(2.8)} \texttt{zero}\ \textsf{odd},$$

because there is no derivation of the right-hand side when the left-hand side is added as an axiom to rules (2.8). Yet the corresponding admissibility judgment

$$\texttt{succ}(\texttt{zero})\ \textsf{even} \models_{(2.8)} \texttt{zero}\ \textsf{odd}$$

is valid, because the hypothesis is false: there is no derivation of $\texttt{succ}(\texttt{zero})$ even from rules (2.8). Even so, the derivability

$$\texttt{succ}(\texttt{zero})\ \mathsf{even} \vdash_{(2.8)} \texttt{succ}(\texttt{succ}(\texttt{zero}))\ \mathsf{odd}$$

is valid, because we may derive the right-hand side from the left-hand side by composing rules (2.8).

Evidence for admissibility can be thought of as a mathematical function transforming derivations $\triangledown_1, \ldots, \triangledown_n$ of the hypotheses into a derivation $\triangledown$ of the consequent. Therefore, the admissibility judgment enjoys the same structural properties as derivability and hence is a form of hypothetical judgment:

Reflexivity If J is derivable from the original rules, then J is derivable from the original rules: $J \models_{\mathcal{R}} J$.

Weakening If J is derivable from the original rules assuming that each of the judgments in Γ are derivable from these rules, then J must also be derivable assuming that Γ and K are derivable from the original rules: if $\Gamma \models_{\mathcal{R}} J$, then $\Gamma, K \models_{\mathcal{R}} J$.

Transitivity If $\Gamma, K \models_{\mathcal{R}} J$ and $\Gamma \models_{\mathcal{R}} K$, then $\Gamma \models_{\mathcal{R}} J$. If the judgments in Γ are derivable, so is K, by assumption, and hence so are the judgments in Γ, K, and hence so is J.

Theorem 3.3. *The admissibility judgment $\Gamma \models_{\mathcal{R}} J$ enjoys the structural properties of entailment.*

Proof Follows immediately from the definition of admissibility as stating that if the hypotheses are derivable relative to $\mathcal{R}$, then so is the conclusion. □

If a rule r is admissible with respect to a rule set $\mathcal{R}$, then $\vdash_{\mathcal{R},r} J$ is equivalent to $\vdash_{\mathcal{R}} J$. For if $\vdash_{\mathcal{R}} J$, then obviously $\vdash_{\mathcal{R},r} J$, by simply disregarding r. Conversely, if $\vdash_{\mathcal{R},r} J$, then we may replace any use of r by its expansion in terms of the rules in $\mathcal{R}$. It follows by rule induction on $\mathcal{R}, r$ that every derivation from the expanded set of rules $\mathcal{R}, r$ can be transformed into a derivation from $\mathcal{R}$ alone. Consequently, if we wish to prove a property of the judgments derivable from $\mathcal{R}, r$, when r is admissible with respect to $\mathcal{R}$, it suffices show that the property is closed under rules $\mathcal{R}$ alone, because its admissibility states that the consequences of rule r are implicit in those of rules $\mathcal{R}$.

3.2 Hypothetical Inductive Definitions

It is useful to enrich the concept of an inductive definition to allow rules with derivability judgments as premises and conclusions. Doing so lets us introduce *local hypotheses* that apply only in the derivation of a particular premise, and also allows us to constrain inferences based on the *global hypotheses* in effect at the point where the rule is applied.

A *hypothetical inductive definition* consists of a set of *hypothetical rules* of the following form:

$$\frac{\Gamma\,\Gamma_1 \vdash J_1 \quad \dots \quad \Gamma\,\Gamma_n \vdash J_n}{\Gamma \vdash J} \,. \tag{3.9}$$

The hypotheses Γ are the *global hypotheses* of the rule, and the hypotheses Γ_i are the *local hypotheses* of the ith premise of the rule. Informally, this rule states that J is a derivable consequence of Γ when each J_i is a derivable consequence of Γ, augmented with the hypotheses Γ_i. Thus, one way to show that J is derivable from Γ is to show, in turn, that each J_i is derivable from $\Gamma\,\Gamma_i$. The derivation of each premise involves a "context switch" in which we extend the global hypotheses with the local hypotheses of that premise, establishing a new set of global hypotheses for use within that derivation.

We require that all rules in a hypothetical inductive definition be *uniform* in the sense that they are applicable in *all* global contexts. Uniformity ensures that a rule can be presented in *implicit*, or *local form*,

$$\frac{\Gamma_1 \vdash J_1 \quad \dots \quad \Gamma_n \vdash J_n}{J} \,, \tag{3.10}$$

in which the global context has been suppressed with the understanding that the rule applies for any choice of global hypotheses.

A hypothetical inductive definition is to be regarded as an ordinary inductive definition of a *formal derivability judgment* $\Gamma \vdash J$ consisting of a finite set of basic judgments Γ and a basic judgment J. A set of hypothetical rules $\mathcal{R}$ defines the strongest formal derivability judgment that is *structural* and *closed* under uniform rules $\mathcal{R}$. Structurality means that the formal derivability judgment must be closed under the following rules:

$$\frac{}{\Gamma, J \vdash J} \tag{3.11a}$$

$$\frac{\Gamma \vdash J}{\Gamma, K \vdash J} \tag{3.11b}$$

$$\frac{\Gamma \vdash K \quad \Gamma, K \vdash J}{\Gamma \vdash J} \tag{3.11c}$$

These rules ensure that formal derivability behaves like a hypothetical judgment. We write $\Gamma \vdash_{\mathcal{R}} J$ to mean that $\Gamma \vdash J$ is derivable from rules $\mathcal{R}$.

The principle of *hypothetical rule induction* is just the principle of rule induction applied to the formal hypothetical judgment. So to show that $\mathcal{P}(\Gamma \vdash J)$ when $\Gamma \vdash_{\mathcal{R}} J$, it is enough to show that $\mathcal{P}$ is closed under the rules of $\mathcal{R}$ and under the structural rules.[1] Thus, for each rule of the form (3.9), whether structural or in $\mathcal{R}$, we must show that

$$\text{if } \mathcal{P}(\Gamma\,\Gamma_1 \vdash J_1) \text{ and } \dots \text{ and } \mathcal{P}(\Gamma\,\Gamma_n \vdash J_n),\ \text{then } \mathcal{P}(\Gamma \vdash J).$$

But this is just a restatement of the principle of rule induction given in Chapter 2, specialized to the formal derivability judgment $\Gamma \vdash J$.

In practice, we usually dispense with the structural rules by the method described in Section 3.1.2. By proving that the structural rules are admissible, any proof by rule induction

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may restrict attention to the rules in $\mathcal{R}$ alone. If all rules of a hypothetical inductive definition are uniform, the structural rules (3.11b) and (3.11c) are clearly admissible. Usually, rule (3.11a) must be postulated explicitly as a rule, rather than shown to be admissible on the basis of the other rules.

3.3 General Judgments

General judgments codify the rules for handling variables in a judgment. As in mathematics in general, a variable is treated as an *unknown*, ranging over a specified set of objects. A *generic* judgment states that a judgment holds for any choice of objects replacing designated variables in the judgment. Another form of general judgment codifies the handling of symbolic parameters. A *parametric* judgment expresses generality over any choice of fresh renamings of designated symbols of a judgment. To keep track of the active variables and symbols in a derivation, we write $\Gamma \vdash_{\mathcal{R}}^{\mathcal{U};\mathcal{X}} J$ to say that J is derivable from Γ according to rules $\mathcal{R}$, with objects consisting of abt's over symbols $\mathcal{U}$ and variables $\mathcal{X}$.

The concept of uniformity of a rule must be extended to require that rules be *closed under renaming and substitution* for variables and *closed under renaming* for parameters. More precisely, if $\mathcal{R}$ is a set of rules containing a free variable x of sort s, then it must also contain all possible substitution instances of abt's a of sort s for x, including those that contain other free variables. Similarly, if $\mathcal{R}$ contains rules with a parameter u, then it must contain all instances of that rule obtained by renaming u of a sort to any u' of the same sort. Uniformity rules out stating a rule for a variable, without also stating it for all instances of that variable. It also rules out stating a rule for a parameter without stating it for all possible renamings of that parameter.

Generic derivability judgment is defined by

$$\mathcal{Y} \mid \Gamma \vdash_{\mathcal{R}}^{\mathcal{X}} J \quad \text{iff} \quad \Gamma \vdash_{\mathcal{R}}^{\mathcal{X}\mathcal{Y}} J,$$

where $\mathcal{Y} \cap \mathcal{X} = \emptyset$. Evidence for generic derivability consists of a *generic derivation* $\triangledown$ involving the variables $\mathcal{X}\,\mathcal{Y}$. So long as the rules are uniform, the choice of $\mathcal{Y}$ does not matter, in a sense to be explained shortly.

For example, the generic derivation $\triangledown$,

$$\dfrac{\dfrac{\overline{x \ \mathsf{nat}}}{\mathtt{succ}(x) \ \mathsf{nat}}}{\mathtt{succ}(\mathtt{succ}(x)) \ \mathsf{nat}},$$

is evidence for the judgment

$$x \mid x \ \mathsf{nat} \vdash_{(2.2)}^{\mathcal{X}} \mathtt{succ}(\mathtt{succ}(x)) \ \mathsf{nat}$$

provided $x \notin \mathcal{X}$. Any other choice of x would work just as well, as long as all rules are uniform.

The generic derivability judgment enjoys the following *structural properties* governing the behavior of variables, provided that $\mathcal{R}$ is uniform.

Proliferation If $\mathcal{Y} \mid \Gamma \vdash^{\mathcal{X}}_{\mathcal{R}} J$, then $\mathcal{Y}, y \mid \Gamma \vdash^{\mathcal{X}}_{\mathcal{R}} J$.
Renaming If $\mathcal{Y}, y \mid \Gamma \vdash^{\mathcal{X}}_{\mathcal{R}} J$, then $\mathcal{Y}, y' \mid [y \leftrightarrow y']\Gamma \vdash^{\mathcal{X}}_{\mathcal{R}} [y \leftrightarrow y']J$ for any $y' \notin \mathcal{X}\,\mathcal{Y}$.
Substitution If $\mathcal{Y}, y \mid \Gamma \vdash^{\mathcal{X}}_{\mathcal{R}} J$ and $a \in \mathcal{B}[\mathcal{X}\,\mathcal{Y}]$, then $\mathcal{Y} \mid [a/y]\Gamma \vdash^{\mathcal{X}}_{\mathcal{R}} [a/y]J$.

Proliferation is guaranteed by the interpretation of rule schemes as ranging over all expansions of the universe. Renaming is built into the meaning of the generic judgment. It is left implicit in the principle of substitution that the substituting abt is of the same sort as the substituted variable.

Parametric derivability is defined analogously to generic derivability, albeit by generalizing over symbols, rather than variables. Parametric derivability is defined by

$$\mathcal{V} \parallel \mathcal{Y} \mid \Gamma \vdash^{\mathcal{U};\mathcal{X}}_{\mathcal{R}} J \quad \text{iff} \quad \mathcal{Y} \mid \Gamma \vdash^{\mathcal{U}\,\mathcal{V};\mathcal{X}}_{\mathcal{R}} J,$$

where $\mathcal{V} \cap \mathcal{U} = \emptyset$. Evidence for parametric derivability consists of a derivation $\triangledown$ involving the symbols $\mathcal{V}$. Uniformity of $\mathcal{R}$ ensures that any choice of parameter names is as good as any other; derivability is stable under renaming.

3.4 Generic Inductive Definitions

A *generic inductive definition* admits generic hypothetical judgments in the premises of rules, with the effect of augmenting the variables, as well as the rules, within those premises. A *generic rule* has the form

$$\frac{\mathcal{Y}\,\mathcal{Y}_1 \mid \Gamma\,\Gamma_1 \vdash J_1 \quad \dots \quad \mathcal{Y}\,\mathcal{Y}_n \mid \Gamma\,\Gamma_n \vdash J_n}{\mathcal{Y} \mid \Gamma \vdash J}. \tag{3.12}$$

The variables $\mathcal{Y}$ are the *global variables* of the inference, and, for each $1 \le i \le n$, the variables $\mathcal{Y}_i$ are the *local variables* of the ith premise. In most cases, a rule is stated for *all* choices of global variables and global hypotheses. Such rules can be given in *implicit form*,

$$\frac{\mathcal{Y}_1 \mid \Gamma_1 \vdash J_1 \quad \dots \quad \mathcal{Y}_n \mid \Gamma_n \vdash J_n}{J}. \tag{3.13}$$

A generic inductive definition is just an ordinary inductive definition of a family of *formal generic judgments* of the form $\mathcal{Y} \mid \Gamma \vdash J$. Formal generic judgments are identified up to renaming of variables, so that the latter judgment is treated as identical to the judgment $\mathcal{Y}' \mid \widehat{\rho}(\Gamma) \vdash \widehat{\rho}(J)$ for any renaming $\rho : \mathcal{Y} \leftrightarrow \mathcal{Y}'$. If $\mathcal{R}$ is a collection of generic rules, we write $\mathcal{Y} \mid \Gamma \vdash_{\mathcal{R}} J$ to mean that the formal generic judgment $\mathcal{Y} \mid \Gamma \vdash J$ is derivable from rules $\mathcal{R}$.

When specialized to a set of generic rules, the principle of rule induction states that to show $\mathcal{P}(\mathcal{Y} \mid \Gamma \vdash J)$ when $\mathcal{Y} \mid \Gamma \vdash_{\mathcal{R}} J$, it is enough to show that $\mathcal{P}$ is closed under the rules

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$\mathcal{R}$. Specifically, for each rule in $\mathcal{R}$ of the form (3.12), we must show that

$$\text{if } \mathcal{P}(\mathcal{Y}\,\mathcal{Y}_1 \mid \Gamma\,\Gamma_1 \vdash J_1)\ \ldots\ \mathcal{P}(\mathcal{Y}\,\mathcal{Y}_n \mid \Gamma\,\Gamma_n \vdash J_n) \text{ then } \mathcal{P}(\mathcal{Y} \mid \Gamma \vdash J).$$

By the identification convention (stated in Chapter 1), the property $\mathcal{P}$ must respect renamings of the variables in a formal generic judgment.

To ensure that the formal generic judgment behaves like a generic judgment, we must always ensure that the following *structural rules* are admissible:

$$\frac{}{\mathcal{Y} \mid \Gamma, J \vdash J} \tag{3.14a}$$

$$\frac{\mathcal{Y} \mid \Gamma \vdash J}{\mathcal{Y} \mid \Gamma, J' \vdash J} \tag{3.14b}$$

$$\frac{\mathcal{Y} \mid \Gamma \vdash J}{\mathcal{Y}, x \mid \Gamma \vdash J} \tag{3.14c}$$

$$\frac{\mathcal{Y}, x' \mid [x \leftrightarrow x']\Gamma \vdash [x \leftrightarrow x']J}{\mathcal{Y}, x \mid \Gamma \vdash J} \tag{3.14d}$$

$$\frac{\mathcal{Y} \mid \Gamma \vdash J \quad \mathcal{Y} \mid \Gamma, J \vdash J'}{\mathcal{Y} \mid \Gamma \vdash J'} \tag{3.14e}$$

$$\frac{\mathcal{Y}, x \mid \Gamma \vdash J \quad a \in \mathcal{B}[\mathcal{Y}]}{\mathcal{Y} \mid [a/x]\Gamma \vdash [a/x]J} \tag{3.14f}$$

The admissibility of rule (3.14a) is, in practice, ensured by explicitly including it. The admissibility of rules (3.14b) and (3.14c) is assured if each of the generic rules is uniform, because we may assimilate the added variable x to the global variables, and the added hypothesis J, to the global hypotheses. The admissibility of rule (3.14d) is ensured by the identification convention for the formal generic judgment. Rule (3.14f) must be verified explicitly for each inductive definition.

The concept of a generic inductive definition extends to parametric judgments as well. Briefly, rules are defined on formal parametric judgments of the form $\mathcal{V} \parallel \mathcal{Y} \mid \Gamma \vdash J$, with symbols $\mathcal{V}$, as well as variables, $\mathcal{Y}$. Such formal judgments are identified up to renaming of its variables and its symbols to ensure that the meaning is independent of the choice of variable and symbol names.

3.5 Notes

The concepts of entailment and generality are fundamental to logic and programming languages. The formulation given here builds on Martin-Löf (1983, 1987) and Avron (1991). Hypothetical and general reasoning are consolidated into a single concept in the AUTOMATH languages (Nederpelt et al., 1994) and in the LF Logical Framework (Harper et al., 1993). These systems allow arbitrarily nested combinations of hypothetical and general judgments, whereas the present account considers only general hypothetical judgments over basic judgment forms. On the other hand, we consider here symbols, as well as variables, which are not present in these previous accounts. Parametric judgments are required for specifying languages that admit the dynamic creation of "new" objects (see Chapter 34).

Exercises

3.1. *Combinators* are inductively defined by the rule set $\mathcal{C}$ given as follows:

$$\frac{}{\texttt{s}\ \mathsf{comb}} \tag{3.15a}$$

$$\frac{}{\texttt{k}\ \mathsf{comb}} \tag{3.15b}$$

$$\frac{a_1\ \mathsf{comb} \quad a_2\ \mathsf{comb}}{\mathtt{ap}(a_1;a_2)\ \mathsf{comb}} \tag{3.15c}$$

Give an inductive definition of the *length* of a combinator defined as the number of occurrences of S and K within it.

3.2. The general judgment

$$x_1, \ldots, x_n \mid x_1\ \mathsf{comb}, \ldots, x_n\ \mathsf{comb} \vdash_{\mathcal{C}} A\ \mathsf{comb}$$

states that A is a combinator that may involve the variables $x_1, \ldots, x_n$. Prove that if $x \mid x\ \mathsf{comb} \vdash_{\mathcal{C}} a_2\ \mathsf{comb}$ and $a_1\ \mathsf{comb}$, then $[a_1/x]a_2\ \mathsf{comb}$ by induction on the derivation of the first hypothesis of the implication.

3.3. *Conversion*, or *equivalence*, of combinators is expressed by the judgment $A \equiv B$ defined by the rule set $\mathcal{E}$ extending $\mathcal{C}$ as follows:[2]

$$\frac{a\ \mathsf{comb}}{a \equiv a} \tag{3.16a}$$

$$\frac{a_2 \equiv a_1}{a_1 \equiv a_2} \tag{3.16b}$$

$$\frac{a_1 \equiv a_2 \quad a_2 \equiv a_3}{a_1 \equiv a_3} \tag{3.16c}$$

$$\frac{a_1 \equiv a_1' \quad a_2 \equiv a_2'}{a_1\, a_2 \equiv a_1'\, a_2'} \tag{3.16d}$$

$$\frac{a_1\ \mathsf{comb} \quad a_2\ \mathsf{comb}}{\texttt{k}\, a_1\, a_2 \equiv a_1} \tag{3.16e}$$

$$\frac{a_1\ \mathsf{comb} \quad a_2\ \mathsf{comb} \quad a_3\ \mathsf{comb}}{\texttt{s}\, a_1\, a_2\, a_3 \equiv (a_1\, a_3)\, (a_2\, a_3)} \tag{3.16f}$$

The no-doubt mysterious motivation for the last two equations will become clearer in a moment. For now, show that

$$x \mid x\ \mathsf{comb} \vdash_{\mathcal{C} \cup \mathcal{E}} \texttt{s}\,\texttt{k}\,\texttt{k}\,x \equiv x.$$

3.4. Show that if $x \mid x \text{ comb} \vdash_{\mathcal{C}} a \text{ comb}$, then there is a combinator a', written $[x]\,a$ and called *bracket abstraction*, such that

$$x \mid x \text{ comb} \vdash_{\mathcal{C}\cup\mathcal{E}} a'\,x \equiv a.$$

Consequently, by Exercise **3.2**, if a'' comb, then

$$([x]\,a)\,a'' \equiv [a''/x]a.$$

Hint: Inductively define the judgment

$$x \mid x \text{ comb} \vdash \mathsf{abs}_x\ a \text{ is } a',$$

where $x \mid x \text{ comb} \vdash a \text{ comb}$. Then argue that it defines a' as a binary function of x and a. The motivation for the conversion axioms governing $\mathtt{k}$ and $\mathtt{s}$ should become clear while developing the proof of the desired equivalence.

3.5. Prove that bracket abstraction, as defined in Exercise **3.4**, is *non-compositional* by exhibiting a and b such that a comb and

$$x\ y \mid x \text{ comb } y \text{ comb} \vdash_{\mathcal{C}} b \text{ comb}$$

such that $[a/y]([x]\,b) \neq [x]\,([a/y]b)$. *Hint*: Consider the case that b is y.

Suggest a modification to the definition of bracket abstraction that is *compositional* by showing under the same conditions given above that

$$[a/y]([x]\,b) = [x]\,([a/y]b).$$

3.6. Consider the set $\mathcal{B}[\mathcal{X}]$ of abt's generated by the operators $\mathtt{ap}$, with arity $(\mathsf{Exp}, \mathsf{Exp})\mathsf{Exp}$, and λ, with arity $(\mathsf{Exp}.\mathsf{Exp})\mathsf{Exp}$, and possibly involving variables in $\mathcal{X}$, all of which are of sort Exp. Give an inductive definition of the judgment b closed, which specifies that b has no free occurrences of the variables in $\mathcal{X}$. *Hint*: it is essential to give an inductive definition of the hypothetical, general judgment

$$x_1, \ldots, x_n \mid x_1 \text{ closed}, \ldots, x_n \text{ closed} \vdash b \text{ closed}$$

in order to account for the binding of a variable by the λ operator. The hypothesis that a variable is closed seems self-contradictory in that a variable obviously occurs free in itself. Explain why this is not the case by examining carefully the meaning of the hypothetical and general judgments.

Notes

1 Writing $\mathcal{P}(\Gamma \vdash J)$ is a mild abuse of notation in which the turnstile is used to separate the two arguments to $\mathcal{P}$ for the sake of readability.

2 The combinator $\mathtt{ap}(a_1;a_2)$ is written $a_1\,a_2$ for short, left-associatively when used in succession.