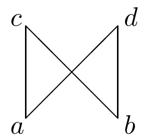
Definition: a **partial order** (sometimes called a *partially ordered set* or *poset*) is a pair, $(D, \sqsubseteq)$ where:
D is a set and
$\sqsubseteq$ is a reflexive, transitive, and anti-symetric relation such that:
$\forall x.\ x \sqsubseteq x$
$\forall x, y, z.\ (x \sqsubseteq y) \wedge (y \sqsubseteq z) \ \Rightarrow\ (x \sqsubseteq z)$
$\forall xy.\ (x \sqsubseteq y) \wedge (y \sqsubseteq x) \ \Rightarrow\ (x = y)$

Definition: Let X be a subset of D, then:
$d \in D$ is an **upper-bound** (or UB) for X iff $\forall x \in X.\ (x \sqsubseteq d)$



c is an *upper-bound* for $\{a, b, c\}$ and
c & d are both *upper-bounds* for $\{a, b\}$

Definition: d is a **least-upper-bound** (or LUB) for X iff:
(1) d is an *upper-bound* for X and
(2) if d' is any *upper-bound* for X then $d \sqsubseteq d'$

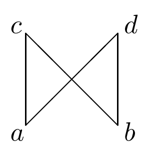
Property: if d & d' are both LUBs of X then $d = d'$

Proof: Since d and d' are LUBs of X, $(d \sqsubseteq d')$ and $(d' \sqsubseteq d)$, $(d = d')$ by *anti-symetry* of $\sqsubseteq$
- If $\sqsubseteq$ is not *anti-symetric* we call it a *pre-order*
- We write $\bigsqcup X$ for the *least-upper-bound* of X, if it exists

What kinds of sets have LUBs ?
(1) no restrition: (*poset*)
(2) Every finite subset has a LUB (*lattice*)
(3) Every subset has a LUB (*complete lattice*)

Somewhere between 1 & 2 is our interesting class of sets

Definition: Given a *poset* $(D, \sqsubseteq)$, $X \subseteq D$ is **directed** iff every finite set $F \subseteq X$ has an *upper-bound* in X



directed	not directed
$\{a, c\}$	$\{a, b\}$
$\{a, b, c\}$	$\{a, b, c, d\}$
singleton sets	
pairs of $\sqsubseteq$	

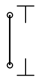
Definition: a **complete-partial-order** (CPO) is a *poset* where every *directed* subset has a LUB .

Definition: a **pointed-CPO** is a CPO with a *least-element*, usually *bottom*, $(\bot)$

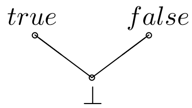
Any finite PO is a CPO since any directed set X is finite, choose $F = X$ so X has an *upper-bound* in X, that must be the LUB .

Interesting *posets*:

Singleton: $\{\cdot\}$, $\mathbf{1}$, or $\mathbf{U}$

2 element: 

or $\mathbf{O}$

3 element: 

, $\mathbf{B}$, or $\mathbf{T}$

$\omega^\top$: $(\mathbf{N} \cup \{\top\})$

If A is any set then $\langle \mathbf{P}(A), \subseteq \rangle$ is a CPO

$X \subseteq \mathbf{P}(A)$, $\bigsqcup X = \bigcup X$

If A is any set then $\langle A, \{(x, y) \mid x = y\} \rangle$ is a CPO (but not *pointed*)

but, $\langle A \cup \{\bot\}, \{(x, y) \mid (x = y) \vee (x = \bot)\} \rangle$ is

---

Definition: **Products of Posets**

$(D, \sqsubseteq_D) \times (E, \sqsubseteq_E) = (D \times E, \sqsubseteq_{D \times E})$

where

$(d, e) \sqsubseteq_{D \times E} (d', e')$ iff $(d \sqsubseteq_D) \wedge (e \sqsubseteq_E e')$

If $P \subseteq D \times E$ is directed then P has a LUB

Claim: Let P be directed, $P_1 = \{x \mid \exists y.(x, y) \in P\}$ and $P_2 = \{y \mid \exists y.(x, y) \in P\}$, then P_1 & P_2 are directed

Proof: if $\{x_1, ..., x_n\} \in P_1$ and $\{y_1, ..., y_n\} \in P_2$ such that $\{(x_1, y_1), ..., (x_n, y_n)\} \subseteq P$ then by the definition of $\sqsubseteq_{P_1 \times P_2}$, since P is directed so are P_1 and P_2

Partial Functions

If S and T are sets then $S \multimap T$ defines the set of **partial-functions** from S to T

$f \sqsubseteq g$ iff $\forall x \in S$, if $f(x)$ is defined then so is $g(x)$, and $f(x) = g(x)$

Alternatively:

$graph(f) = \{(x, y) \mid x \in S \ \wedge \ y = f(x)\}$, then $f \sqsubseteq g$ iff $graph(f) \subseteq graph(f)$

$X \subseteq [S \multimap T]$ is directed iff for any finite $F \subseteq X$, F has a UB in X

Let $X \subseteq S \multimap T$, $F \subset X$ with F finite.
Let $f \in F = \{(x_1, f(x_1)), ..., (x_n, f(x_n))\}$

We want $g : S \to T$ such that $g(x) = f(x)$ if $\exists f \in X$ such that $f(x)$ is defined

Lemma I: if $f_1, f_2 \in X$ and $f_1(x)$ & $f_2(x)$ are defined, then $f_1(x) = f_2(x)$

Proof: $\{f_1, f_2\}$ is a finite subset of X, therefore it has an UB in X,

i.e. $\exists f_3$ such that $f_1 \sqsubseteq f_3$ and $f_2 \sqsubseteq f_3$. So $f_1(x) = f_3(x)$ and $f_2(x) = f_3(x)$

This all means that:

$S \multimap T$ is a CPO

Definition: Let $f : D \to E$ where $D \equiv (D, \sqsubseteq_D)$ and $E \equiv (E, \sqsubseteq_E)$
The function f is **monotone** iff $\forall d, d'$, if $d \sqsubseteq_D d'$ then $f(d) \sqsubseteq_E f(d')$

Lemma (Exchange):
Let D & D' be CPOs, $P \subseteq D$ & $Q \subseteq D'$ be directed, and D'' be any *poset*.
Let $f : D \times D \to D''$ be monotone, then:

$$\bigsqcup_{x \in P} \bigsqcup_{y \in Q} f(x, y) = \bigsqcup_{y \in Q} \bigsqcup_{x \in P} f(x, y)$$

Proof: Must show that:

$$\bigsqcup_{x \in P} \bigsqcup_{y \in Q} f(x, y) \sqsubseteq \bigsqcup_{y \in Q} \bigsqcup_{x \in P} f(x, y) \quad \text{and} \quad \bigsqcup_{y \in Q} \bigsqcup_{x \in P} f(x, y) \sqsubseteq \bigsqcup_{x \in P} \bigsqcup_{y \in Q} f(x, y)$$

First, $\bigsqcup_{x \in P}$ is a *LUB* , so it suffices to show that

$$\bigsqcup_{y \in Q} \bigsqcup_{x \in P} f(x, y) \quad \text{is an } \textit{upper-bound} \text{ for } \quad \{\bigsqcup_{y \in Q} f(x, y) \mid x \in P\}$$

So it remains to show that for any $x \in P$

$$\bigsqcup_{y \in Q} f(x, y) \sqsubseteq \bigsqcup_{y \in Q} \bigsqcup_{x \in P} f(x, y)$$

From there it remains to show that for any $y \in Q$

$$f(x, y) \sqsubseteq \bigsqcup_{y \in Q} \bigsqcup_{x \in P} f(x, y)$$

which is obvious since f is *monotone*
Options:
(1) ... if these *LUB*s exist ...
(2) ... let D'' be a CPO, then these *LUB*s exist ...

Lemma (Diagonal):
Let D & D' be CPOs, $P \subseteq D$ be directed, and $f : D \times D \to D'$ be monotone then:

$$\bigsqcup_{x \in P} \bigsqcup_{y \in P} f(x, y) = \bigsqcup_{x \in P} f(x, x)$$

Proof: Homework Excercize #5

There are more notes here from the discussion the day after...

Definition: a function $f : D \to E$ where D & E are *pointed-CPO*s is **strict** iff it is *bottom-preserving*.

Definition: a function $f : D \to E$ where D & E are *CPO*s is **continuous** iff
$\forall X \subseteq D$ where X is directed, $f(\bigsqcup X) = \bigsqcup_{x \in X} f(x)$.

This leads to the following properties:

(1) If f is *continuous*, then f is *monotone* (continuity $\Rightarrow$ monotonicity)

Let $X = \{a, b\}$ where $a \sqsubseteq b$. Then X is *directed*.

Therefore $f(\bigsqcup X) = \bigsqcup\{f(x) \mid x \in X\}$, So

$f(b) = \bigsqcup\{f(a), f(b)\}$, therefore $f(a) \sqsubseteq f(b)$

(2) If f is *monotone* and D is *finite*, then f is *continuous*

(3) If f is *monotone* and D has no *infinite-increasing-chains*, then f is *continuous*

(4) f is *monotone* $\not\Rightarrow$ f is *continuous*

Example:

$f : \omega^\top \rightarrow \mathbf{O}$ with $f(\top) = \top$ and $f(n) = \bot$

$$\bigsqcup_{x \in X} f(x) = \bot \quad \text{and} \quad f(\bigsqcup_{x \in X} x) = \top$$

f is not *continuous* since $\bigsqcup f(x) \neq f(\bigsqcup x)$

Lemma: If $f : D \rightarrow E$ is continuous and $X \subseteq D$ is directed, then

$$\bigsqcup_{x \in X} f(x) \sqsubseteq f(\bigsqcup_{x \in X} x)$$

Proof: For any $x_0 \in X$, $x_0 \sqsubseteq \bigsqcup X$, since f is *monotone*, $f(x_0) \sqsubseteq \bigsqcup f(x)$

This is true for all $x_0 \in X$, so $\bigsqcup f(x)$ is an UB for $\{f(x) \mid x \in X\}$ and is *least*

Claim: If $f : D \rightarrow E$ and $g : E \rightarrow F$ are *continuous* then $(g \circ f) : D \rightarrow F$ is *continuous*

Proof: Must show that:

$$(g \circ f)(\bigsqcup_{x \in X} x) \sqsubseteq \bigsqcup_{x \in X} (g \circ f)(x)$$

Since X is directed, so is $\{f(x) \mid x \in X\}$ by the above Lemma.

$$\begin{aligned} g(f(\bigsqcup X)) &= g(\bigsqcup\{\, f(x) \mid x \in X\}) && : \text{by continuity of } f \\ &= \bigsqcup\{\, g(f(x)) \mid x \in X\} && : \text{by continuity of } g \\ &= \bigsqcup (g \circ f)(x) \end{aligned}$$

Definition: $[D \to E]$ is the set of all *continuous* functions from $D \to E$ such that

$$f \sqsubseteq_{D \to E} g \text{ iff } \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

For some S and T

$S \multimap T$ is a *partial-function-order*
$S \to T_\perp$ is a one to one correspondence by replacing *undefined* by $\perp$

Let $\varphi : (S \to T_\perp) \to (S \multimap T)$ be a function which converts a (total?) function from $S \to T_\perp$ to the corresponding partial function, such that $\varphi(f) = g$ where:

$g(x)$ is undefined if $f(x) = \perp$
$g(x) = f(x)$ otherwise

Claim: $f \sqsubseteq_{S \to T_\perp} g$ iff $\varphi(f) \sqsubseteq \varphi(g)$

Proof: $\forall s \in S.\ (f(s) \sqsubseteq_{T_\perp} g(s)) \iff f(s) = g(s) \lor f(s) = \perp$
So, either $\varphi(f(s)) = g(s)$ or $\varphi(f(s))$ is undefined

Property: $[D \to E]$ is a CPO
Let $P \subseteq [D \to E]$ be *directed*
Define $g : D \to E$ such that $g(x) = \bigsqcup\{\ f(x) \mid f \in P\}$

Claim: $g = \bigsqcup P$

Proof: We must show that:
(1) g is *continuous*,
(2) $\forall f \in P.\ f \sqsubseteq g$, and
(3) $\forall f' \in ub(P).\ f' \sqsubseteq g$

?? I'm not sure where this section fits, my notes might be out of order...

Claim: If P is *directed* then $Q = \{\ f(d) \mid f \in P\}$ is *directed*

Proof: Let $f_1, f_2 \in P$ with $f_1(d), f_2(d) \in Q$
If $f_1, f_2 \in P$ then that share an UB, say $f_3 \in P$ therefore $f_1(d)$ and $f_2(d)$ have an UB, $f_3(d) \in Q$.

So, Q is *directed* and g (from above?) is defined.

Let $Q \subseteq D$ be *directed*. We want to show that:

$$g(\bigsqcup_{x \in Q} x) = \bigsqcup_{x \in Q} g(x)$$

So:

$$\begin{aligned} g(\bigsqcup_{x \in Q} x) &= \bigsqcup_{f \in P} \big(f(\bigsqcup_{x \in Q} x)\big) && \text{by definition of } g \\ &= \bigsqcup_{f \in P} \big(\bigsqcup_{x \in Q} f(x)\big) && \text{since } f \text{ is continuous} \\ &= \bigsqcup_{x \in Q} \big(\bigsqcup_{f \in P} f(x)\big) && \text{by the Exchange lemma} \end{aligned}$$

Reminder:

Continuous $\equiv f(\bigsqcup X) = \bigsqcup_{x \in X} f(x)$
- $\lambda x.x$ is *continuous*
- $D \rightarrow E$, $\{(d, e_0) \mid d \in D\}$ is *continuous*
- $E \rightarrow [D \rightarrow E]$ for any D, E is *continuous*

Definition: $K_{d,e} = \lambda e.\lambda d.e$

If $X \subseteq E$ is *directed* then:

$$K(\bigsqcup X) = \bigsqcup_{e \in X} K(e) = \bigsqcup_{e \in X} (\lambda d.e)$$
$$\bigsqcup_{e \in X} K(e)(d) = \bigsqcup_{e \in X} (e) = \bigsqcup X$$
$$\bigsqcup_{e \in X} \lambda e.\lambda d.e = \bigsqcup_{e \in X} (e) = \bigsqcup X$$

Definition: $S(f)(g)(x) = (f(x))(g(x))$
$x : D$
$g : D \rightarrow E$
$f : D \rightarrow [E \rightarrow F]$
$S : [[D \rightarrow [E \rightarrow F]] \rightarrow [D \rightarrow E] \rightarrow D] \rightarrow F$

Claim: S is *continuous*

Proof: Homework Excercize #6

Must show that if f & g are *continuous* then F is (*continuous* ?)
- Requires three results...

$S : [[[D \Rightarrow [D \Rightarrow F]] \Rightarrow [D \Rightarrow E] \Rightarrow D] \Rightarrow F]$
$K : [D \Rightarrow [D \Rightarrow E]]$

Application:

$$\frac{D \Rightarrow E \quad D}{E}$$

These are complete for propositional logic of *pure-implication*

Definition: Terms
$t ::= x \mid K \mid S \mid (t_1\ t_2)$
$S = \lambda f.\lambda g.\lambda x.((f\ x)(g\ x))$
$K = \lambda x.\lambda y.x$

Claim: Any lambda expression is eqivalent to some combinatory term $\lceil e \rceil$

(1) $\lceil x \rceil = x$
(2) $\lceil e_1\ e_2 \rceil = (\lceil e_1 \rceil\ \lceil e_2 \rceil)$
(3) $\lceil \lambda x.e \rceil = [x](\lceil e \rceil)$
where $[x]$ is an operator on combinatory terms: (*Curry* bracket abstraction)
$[x]x = ((S\ K)\ K)\ x$
$[x]t = (K\ t)$
$[x](t_1\ t_2) = S([x]t_1)([x]t_2) = \lambda x.(([x]t_1)([x]t_1)) = \lambda x.(t_1\ t_2)$

Conclusion... continuous functions are closed under composition