Lecture Outline:

- Multicommodity flow and Sparset cut

1 Multicommodity flow

Demands multicommodity flow: Given graph $G = (V, E)$, edge capacity function $C : E \to \mathbb{Z}^+$. There are $k \geq 1$ commodities, each with its own source s_i, sink t_i, and demand $dem(i)$. The objective is to maximize f such that we can send $f \cdot dem(i)$ units of commodity i from s_i to t_i for each i **simultaneously**, without violating the capacity constraint of any edge.

Sum-flow multicommodity flow: Given graph $G = (V, E)$, edge capacity function $C : E \to \mathbb{Z}^+$. There are $k \geq 1$ commodities, each with its own source s_i, sink t_i. The objective is to maximize the sum of the flow sent from s_i to t_i, over all i, without violating the capacity constraint for any edge.

2 Two examples where Max-Flow is not equal to Min-Cut

It is well known that Max-Flow is equal to Min-Cut in Single Commodity Flow problem. But this is not true for Multicommodity Flow when the number of commodities is greater than 2.

First we give the definition of Min-Cut in multicommodity flow problem.

Definition 1. For any cut $\langle S, \bar{S} \rangle$ of the graph, let $C(S, \bar{S}) = \sum_{e \in \langle S, \bar{S} \rangle} C(e)$ which is the total capacities across this cut, and $D(S, \bar{S}) = \sum_{\{i | s_i \in S \wedge t_i \in \bar{S} \text{ or } s_i \in \bar{S} \wedge t_i \in S\}} dem(i)$ which is the total demand across this cut. Define the Min-Cut as $\eta = \min_{S \subseteq V} \dfrac{C(S, \bar{S})}{D(S, \bar{S})}$. We refer to $\dfrac{C(S, \bar{S})}{D(S, \bar{S})}$ as the *ratio* of cut $(S, \bar{S})$.

Let f^* be the optimal value for demands multicommodity flow. It is clear that $f^* \leq \eta$. The first example (Figure 1, taken from Jon Kleinberg's lecture notes) shows f^* could be strictly smaller than η in multicommodity flow problem.

In the graph, there are 4 flow pairs, each with a demand of 1, and the shortest path between each pair is 2 hops. So the total capacity consumed when we send f^* flow for each commodity is $8f^*$. And there are only 6 edges in the graph. So we have $f^* \leq 3/4$.

The second example gives an even worse ratio between Max-Flow and Min-Cut, where $f^* \leq O\left(\frac{\eta}{\log n}\right)$. This example makes use of Uniform Multicommodity Flow and 3-regular expander graph.

3-regular expanders: 3-regular expander graph has the following properties:

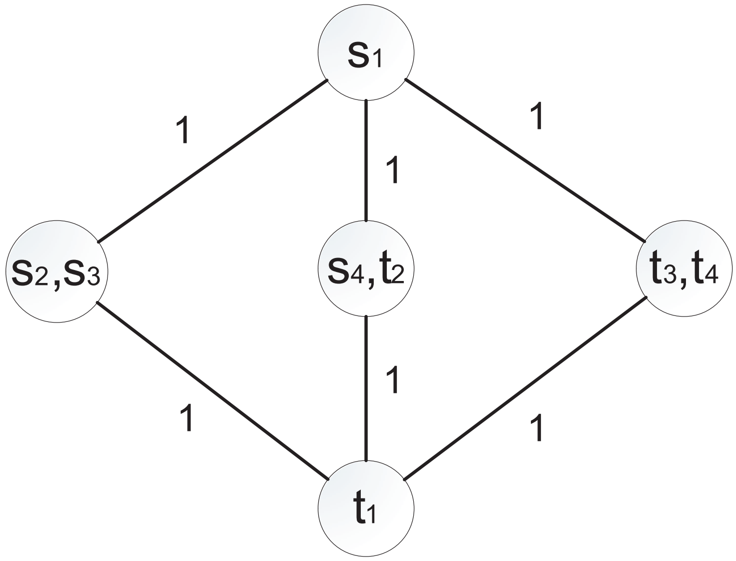

Figure 1: Example of Max-Flow and Min-Cut in multicommodity flow

- degree of every vertex is equal to 3
- $\exists c > 0$ (c is a constant), $\forall S \subseteq V$ if $|S| \leq \dfrac{|V|}{2}$ then $\delta(S) \geq c|S|$. Here $\delta(S)$ is the number of edges that cross cut $\langle S, \bar{S} \rangle$.

Now construct the multicommodity flow problem in the following way. Given a 3-regular expander graph, set the cost of each edge to one, $C(e) = 1$. For each pair of vertices (u, v) set a source and sink pair (s_i, t_i). The demand of each (s_i, t_i) is equal to one, $d_i = 1$.

Theorem 1. $f^* \leq O\left(\dfrac{\eta}{\log n}\right)$

Proof. We first show that $\eta = \Omega(1/n)$. Consider any cut $(S, \bar{S})$. Without loss of generality, we assume $|S| \leq n/2$. Owing to the expansion property, the number of edges crossing the cut is at least $c|S|$. Therefore, the ratio for $(S, \bar{\bar{S}})$ is at least $c|S|/(|S| \cdot |\bar{S}|)$, which is at least $c/n = \Omega(1/n)$.

For each vertex $u \in V$, the number of vertices that are 1-hop away from u is 3 (this is a 3-regular expander graph), the number of vertices that are 2-hop away from u is at most 9, the number of vertices that are 3-hop away from u is at most 27 .... So there are at least $\frac{2n}{3}$ vertices that are more than $\lfloor \log_3 n \rfloor - 1$ hops away from u. And the number of pairs that are separated by more than $\lfloor \log_3 n \rfloor - 1$ hops is at least $n \times \frac{2n}{3} = \frac{2n^2}{3}$. So the total capacity consumed by flows is at least $\frac{2n^2}{3} \times \log_3 n \times f^*$. The total number of edges in this graph is $\frac{3n}{2}$. From

$$\frac{3n}{2} \geq \frac{2n^2}{3} \times \log_3 n \times f^*$$

we have

$$f^* \leq \frac{9}{4n \log_3 n} = O\left(\frac{\eta}{\log n}\right)$$

□

In other words, the Max-Flow for the Uniform Multicommodity Flow problem is at least a $O\left(\frac{\eta}{\log n}\right)$-factor smaller than the min-cut.

3 LP of Demands Multicommodity Flow

Let P_i be the set of paths between pair (s_i, t_i), p^i_j be the j^{th} path in P_i, $dem(i)$ be the demands of pair (s_i, t_i), and f^i_j be the amount of commodity i sent on path p^i_j. Then we get the following LP for Demands Multicommodity Flow problem.

$$\begin{aligned} &\max f \\ \text{s.t.} &\textstyle\sum_{p^i_j} f^i_j \geq f \cdot dem(i) \quad \forall i \\ &\textstyle\sum_{p^i_j : e \in p^i_j} f^i_j \leq c_e \quad \forall e \\ &p^i_j \geq 0 \quad \forall i, j \end{aligned}$$

And its dual is

$$\begin{aligned} &\min \textstyle\sum_e c_e d_e \\ \text{s.t.} &\textstyle\sum_{e \in p^i_j} d_e \geq l_i \quad \forall i, j \\ &\textstyle\sum_i l_i \cdot dem(i) \geq 1 \\ &d_e \geq 0 \quad \forall e \end{aligned}$$

In the dual, d_e can be viewed as the distance assigned to the edge. And l_i is the length of the shortest path between s_i and t_i, according to the distances given by d_e. A feasible solution to the dual yields a lower bound on f^* (by weak duality). In fact, the proof of Theorem **??** can be seen as one based on weak duality. Take the 3-regular expander graph. We saw that $\eta \geq \frac{c}{n}$ where c is a constant. We now show that $f^* \leq O\left(\frac{1}{n \log n}\right)$, using the dual LP defined above. Set $d_e = 2/n^2 \log n$. Then

$$\sum_i l_i \geq \frac{2}{3} n^2 \log n \cdot \frac{2}{n^2 \log n} \geq 1$$

because for each vertex, there are at least $\frac{2n}{3}$ vertices that are more than $\lfloor \log_3 n \rfloor - 1$ hops away from it, and the demand for each pair is 1, $dem(i) = 1$.

By weak duality, we thus have

$$f^* \leq \sum_e d_e = \frac{2}{n^2 \log n} \cdot \frac{3n}{2} = \frac{3}{n \log n} = O\left(\frac{1}{n \log n}\right)$$