You can turn in handwritten solutions to this part of the assignment. Please write clearly and use standard-sized (8.5 by 11in) paper. Solutions should be submitted at the beginning of class on the due date. If you choose to typeset your solutions using LaTeX, you may find the mathpartir.sty package useful.

Problem 1 Below is the syntax, call-by-value operational semantics, and typing rules for the simply-typed λ-calculus with booleans and integers.

Types	τ	$::=$	$\mathsf{Bool} \mid \mathsf{Int} \mid \tau_1 \to \tau_2$
Terms	e	$::=$	$x \mid \lambda x\!:\!\tau.\,e \mid e_1\,e_2 \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{if}\,e\,\mathtt{then}\,e_1\,\mathtt{else}\,e_2 \mid n \mid e_1 < e_2$
Values	v	$::=$	$\mathtt{true} \mid \mathtt{false} \mid n \mid \lambda x\!:\!\tau.\,e$
Evaluation contexts	E	$::=$	$[\cdot] \mid E\,e \mid v\,E \mid \mathtt{if}\,E\,\mathtt{then}\,e_1\,\mathtt{else}\,e_2 \mid E < e_2 \mid v_1 < E$

Evaluation rules:

$$
\begin{aligned}
E[\lambda x\!:\!\tau.\,e\ v] &\longrightarrow E[e[v/x]] && \text{(E-Beta)}\\
E[\mathtt{if\ true\ then}\ e_1\ \mathtt{else}\ e_2] &\longrightarrow E[e_1] && \text{(E-IfTrue)}\\
E[\mathtt{if\ false\ then}\ e_1\ \mathtt{else}\ e_2] &\longrightarrow E[e_2] && \text{(E-IfFalse)}\\
E[n_1 < n_2] &\longrightarrow E[\mathtt{true}] && \text{if } n_1 < n_2 \quad\text{(E-<)}\\
E[n_1 < n_2] &\longrightarrow E[\mathtt{false}] && \text{if } n_1 \geq n_2 \quad\text{(E-Not<)}
\end{aligned}
$$

Typing rules:

$$\text{Term environments} \quad \Gamma \quad ::= \quad \cdot \mid \Gamma, x:\tau$$

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{x:\tau \in \Gamma}{\Gamma \vdash x:\tau}\ \text{(T-Var)}
\qquad
\frac{\Gamma, x:\tau_1 \vdash e:\tau_2}{\Gamma \vdash \lambda x\!:\!\tau_1.\,e : \tau_1 \to \tau_2}\ \text{(T-Lam)}
\qquad
\frac{\Gamma \vdash e_1 : \tau_2 \to \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\,e_2 : \tau}\ \text{(T-App)}
$$

$$
\frac{}{\Gamma \vdash \mathtt{true} : \mathsf{Bool}}\ \text{(T-True)}
\qquad\qquad
\frac{}{\Gamma \vdash \mathtt{false} : \mathsf{Bool}}\ \text{(T-False)}
$$

$$
\frac{\Gamma \vdash e : \mathsf{Bool} \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \mathtt{if}\,e\,\mathtt{then}\,e_1\,\mathtt{else}\,e_2 : \tau}\ \text{(T-If)}
$$

$$
\frac{}{\Gamma \vdash n : \mathsf{Int}}\ \text{(T-Int)}
\qquad\qquad
\frac{\Gamma \vdash e_1 : \mathsf{Int} \quad \Gamma \vdash e_2 : \mathsf{Int}}{\Gamma \vdash e_1 < e_2 : \mathsf{Bool}}\ \text{(T-<)}
$$

For each of the following expressions, say if the expression is well typed or not. If it is well typed, provide a typing derivation; if it is not well typed, explain why in no more than 30 words.

1. $\lambda x\!:\!\mathsf{Int}.\,\lambda y\!:\!\mathsf{Bool}.\,\mathtt{if}\,y\,\mathtt{then}\,x\,\mathtt{else}\,y$

2. $\lambda x\!:\!\mathsf{Bool}.\,\lambda y\!:\!\mathsf{Int}.\,\mathtt{if}\,x\,\mathtt{then}\,(y<1)\,\mathtt{else}\,x$

3. $\lambda x\!:\!\mathsf{Bool}.\,\lambda y\!:\!\mathsf{Bool} \to \mathsf{Int}.\,\mathtt{if}\,x\,\mathtt{then}\,(y\ x)\,\mathtt{else}\,y$

4. $\lambda x\!:\!\mathsf{Int}.\,\lambda y\!:\!\mathsf{Int} \to \mathsf{Bool}.\,\mathtt{if}\,(y\ x)\,\mathtt{then}\,(4<x)\,\mathtt{else}\,(y\ 0)$

5. $\lambda x\!:\!\mathsf{Bool}.\,\lambda y\!:\!(\mathsf{Bool} \to \mathsf{Int}) \to \mathsf{Bool}.\,\mathtt{if}\,x\,\mathtt{then}\,y\,\mathtt{else}\,\lambda z\!:\!\mathsf{Bool} \to \mathsf{Int}.\,x$