You can turn in handwritten solutions to this part of the assignment. Please write clearly and use standard-sized (8.5 by 11in) paper. Solutions should be submitted at my office (328 WVH) by 1pm on the due date.

Problem 2: Type Soundness We saw that the simply-typed λ-calculus ($\lambda^{\rightarrow}$) has a sound type system because it preserves types and guarantees progress of well-typed terms. Thus, well-typed terms do not get stuck (i.e., evaluation is *safe*). Let us add pairs to the call-by-value simply-typed λ-calculus.

$$\begin{array}{llll} \textit{Types} & \tau & ::= & \ldots \mid \tau_1 \times \tau_2 \\ \textit{Terms} & e & ::= & \ldots \mid (e_1, e_2) \mid \texttt{fst}\ e \mid \texttt{snd}\ e \\ \textit{Values} & v & ::= & \ldots \mid (v_1, v_2) \end{array}$$

New evaluation rules:

$$\frac{e_1 \longrightarrow e_1'}{(e_1, e_2) \longrightarrow (e_1', e_2)}\ \text{(E-Pair1)} \qquad \frac{e_2 \longrightarrow e_2'}{(v_1, e_2) \longrightarrow (v_1, e_2')}\ \text{(E-Pair2)}$$

$$\frac{e \longrightarrow e'}{\texttt{fst}\ e \longrightarrow \texttt{fst}\ e'}\ \text{(E-Fst)} \qquad \frac{e \longrightarrow e'}{\texttt{snd}\ e \longrightarrow \texttt{snd}\ e'}\ \text{(E-Snd)}$$

$$\frac{}{\texttt{fst}\ (v_1, v_2) \longrightarrow v_1}\ \text{(E-FstPair)} \qquad \frac{}{\texttt{snd}\ (v_1, v_2) \longrightarrow v_2}\ \text{(E-SndPair)}$$

New typing rules:

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}\ \text{(T-Pair)}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{fst}\ e : \tau_1}\ \text{(T-Fst)} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{snd}\ e : \tau_2}\ \text{(T-Snd)}$$

For this problem, you must extend the proofs of progress and preservation for STLC ($\lambda^{\rightarrow}$)—as well as the proofs of lemmas that these rely on—to demonstrate type soundness for this extended language ($\lambda^{\rightarrow\times}$).

(a) State the inversion lemma.

(b) State and prove the canonical forms lemma.

(c) State the permutation and weakening lemmas.

(d) State and prove the substitution lemma.

(e) Prove the progress and preservation lemmas; their statements are as follows:

Lemma (Progress): If $\vdash e : \tau$ then *either* e is a value *or* there exists some e' such that $e \longrightarrow e'$.

Lemma (Preservation): If $\vdash e : \tau$ and $e \longrightarrow e'$, then $\vdash e' : \tau$.

Note: When proving preservation, use induction on the derivation of $e \longrightarrow e'$.

Note: For the proof portions only of parts (b), (d), and (e), you do not need to show the cases involving functions, application, and function types.