Knowledge assumed in this document:

Algebraic definition of even, odd, irrational.

IF-THEN, IFF, $\rightarrow$, $\leftrightarrow$

$\exists$

$\mathbb{Z}$

prime & prime factor

every integer is a product of primes (see document on induction)

CONTRAPOSITIVE

if Greece wins the world cup, I will be happy (forever)

equivalent

if I'm not happy, Greece has not won the world cup

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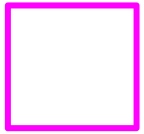
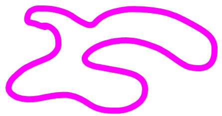
if you are a square, you have corners 

equivalent

if you don't have corners, you are not a square  

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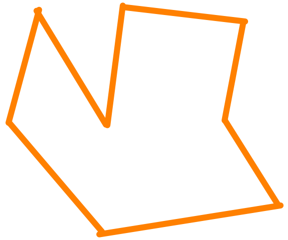
if A then B $\Longleftrightarrow$ if not B, then not A

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \rightarrow B \equiv \neg B \rightarrow \neg A$$

What if $\neg A$ holds, but B is still true?

Greece hasn't won, but I'm still happy



This shape isn't a square, but it has corners

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B \;=\; \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

↳ That's OK; no contradiction. It's not B IFF A

a	b	$a \to b$ valid?	$\neg b$	$\neg a$	$(\neg b) \to (\neg a)$ valid?
T	T	✓	F	F	✓
T	F	✗	T	F	✗
F	T	✓	F	T	✓
F	F	✓	T	T	✓

don't contradict

don't contradict

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x+9$ is even, then x is odd (for $x \in \mathbb{Z}$)

direct

$7x+9 = 2a$ // a: integer $\to$ $7x+9$: even

$x = 2a - 6x - 9$

$x = 2a - 6x - 10 + 1$

$x = 2(a - 3x - 5) + 1$

$x = 2b + 1$ (odd) $(b = a - 3x - 5)$ $\square$

contrapositive

Suppose x is not odd: $x = 2c$

$7x + 9 = 7 \cdot 2c + 9$

$= 14c + 8 + 1$

$= 2 \cdot (7c + 4) + 1$

$= 2 \cdot d + 1$ $(d = 7c + 4)$

$7x + 9 = \text{odd}$ $\square$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

$$x^2 - 6x + 5 = 2a$$

direct

$$x^2 - 6x + (5 - 2a) = 0$$

$$x = \frac{6 \pm \sqrt{36 + 8a - 20}}{2}$$

$$x = 3 \pm \sqrt{4 + 2a}$$

$$\vdots \quad ?$$

$$x = 2b + 1$$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

direct	contrapositive
$x^2 - 6x + 5 = 2a$ $\vdots$? $x = 2b + 1$	Suppose x is not odd: $x = 2c$ $x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$ $= 4c^2 - 12c + 5$ $= 4c^2 - 12c + 4 + 1$ $= 2 \cdot (2c^2 - 6c + 2) + 1$ $= 2 \cdot d + 1$ $(d = 2c^2 - 6c + 2)$ $=$ not even $\square$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove : if x is irrational then $\sqrt{x}$ is irrational

direct	contrapositive
???	Suppose $\sqrt{x}$ is not irrational $\sqrt{x} = \frac{a}{b}$ $\quad a, b \in \mathbb{Z}$ $x = \frac{a^2}{b^2}$: not irrational $\square$

PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

still proving if A then B for now

$(a+b)\cdot(a-b) \longleftrightarrow a^2-ab+ba-b^2 \longleftrightarrow a^2-b^2$

You can prove something directly (in one direction)
or work in both directions

instead of starting w/ $\neg B$ & leading to $\neg A$
(which contradicts $A \to \neg B$)

$$\boxed{\text{assume both } A \text{ and } \neg B \text{ are true \& arrive at some contradicting statement}}$$

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: x is even & x is odd

$$\begin{array}{ccc} x \text{ is even} & & x \text{ is odd} \\ \downarrow & & \downarrow \\ x = 2a & & x = 2b+1 \end{array}$$

(a: int.) (b: int.)

$2a = 2b+1$

$a = b + \frac{1}{2}$

impossible / absurd / contradiction $\square$

Could also plug $b+\frac{1}{2}$ into $x=2a$ & conclude x is odd.

Notice we met halfway at an incorrect statement.

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number s.t. $az + b = 0$.

(if $\underbrace{ax+b=0}_{A}$ then for $y \neq x$, $\underbrace{ay+b \neq 0}_{B}$)

Assume $A \wedge \neg B$: $ax+b=0$ & $ay+b=0$

$$ax+b = ay+b$$

$$ax = ay$$

$$x = y$$ — contradicts $\square$

Prove: if A then B

Assume $A \wedge \neg B$, get contradiction. ✓

Does it work if we assume $\neg A \wedge B$ and get a contradiction?

NO

$A \to B$ tells us nothing about what happens when $\neg A$.

It would work if we were proving $A \leftrightarrow B$

$\sqrt{2}$ IS IRRATIONAL - PROOF BY CONTRADICTION

1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

1) $\nexists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a,b\}$ w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow \boxed{\begin{array}{l} a^2 : \text{even} \\ (a : \text{even}) \end{array}}$

$$(2x+1)\cdot(2x+1) = 4x^2+4x+1$$
$$= 2\cdot(2x^2+2x)+1$$

$a : \text{odd} \rightarrow a^2 : \text{odd}$

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

4) conclude that (2) is false thus the initial claim is true

1) $\nexists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a,b\}$ w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$: even (a: even)

$a = 2c$ $\{c : \text{int}\} \Rightarrow 2b^2 = 4c^2 \Rightarrow b$: even

$\sqrt{2} = \frac{a}{b} = \frac{2c}{2d}$ contradiction

$\square$

THERE ARE AN INFINITE NUMBER OF PRIMES

(proof by contradiction)

- Assume that #primes is finite: $p_1, p_2, \dots, p_n$
- Let $t = 1 + \prod_{i=1}^{n} p_i$ (i.e., $1 + p_1 \times p_2 \times \cdots \times p_n$)
- Notice $t > p_i$ for all i. So if t is prime, contradiction.
- If t is not prime then $\exists$ prime factor $q \neq t$ of t

  t is composite

  - if $q \neq p_i$ (for all i), contradiction.
  - if $q = p_j$, we know q divides $\prod_{i=1}^{n} p_i$

    but then it can't also divide $1 + \prod_{i=1}^{n} p_i$ (contr.) $\square$

SMALLEST COUNTEREXAMPLE

Prove that the first n odd natural numbers sum to n^2.

$$\begin{array}{lcccccccc} i = & 1 & 2 & 3 & 4 & \cdots & (n-1) & n & \\ & 1 & + 3 & + 5 & + 7 & + \cdots & + (2n-3) & + (2n-1) & = n^2 \end{array}$$

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$.

We saw the claim is true for small n.

If the claim is false, there must be some smallest number x $(\leq n)$

for which $\sum_{i=1}^{x} 2i-1 \neq x^2$

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \; ?$$

if false, then $\exists x$ for which it is false & $x-1$ for which it is true

↳ in fact for all $< x$

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i = x-1} = (x-1)^2$$

← either this should have been $\neq$

$$1 + 3 + 5 + \cdots + (2x-3) + \overbrace{(2x-1)}^{i = x} \neq x^2$$

← or this should have been $=$

$$(x-1)^2 + 2x - 1 \neq x^2$$

$$x^2 - \cancel{2x} + \cancel{1} + \cancel{2x} - \cancel{1} \neq x^2$$

contradiction

... which contradicts the smallest counterexample assumption, i.e.,

THERE IS NO (SMALLEST) COUNTEREXAMPLE

↳ CLAIM IS TRUE

SMALLEST COUNTEREXAMPLE

recap

see "well-ordering principle"

- be able to "count" & "order" instances of the claim

  (case / example)

- prove the claim for smallest instance
- assume the claim is false: then there is a smallest instance, E_i, for which it is false

  (smallest counterexample)

- this implies the claim is true for the next smallest instance, E_{i-1}.
- use E_i & E_{i-1} to get a contradiction (to the existence of any counterexample)

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

(notice
$$\begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array}$$
)

- use smallest counterexample

  ($n = 2, 3, 4$ are not counterexamples)

  why can we? → Claim is true for smallest instance ($n=5$)

- assume $\exists$ smallest counterexample x. $\}$ $\boxed{2^x \leq x^2}$ $(x > 5)$

  (& for $y \geq 5$, if $y < x$ then $2^y > y^2$)

- focus on $x-1$: $\boxed{2^{x-1} > (x-1)^2}$ → combine to get contradiction

$$\boxed{2^x \leq x^2}$$

because x is a counterexample

$$\boxed{2^{x-1} > (x-1)^2}$$

because x is the smallest counterexample

and not the smallest case

!

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$2^x > 2x^2 - 4x + 2$$

$$2^x > x^2 + (x^2 - 4x + 2)$$

if $x^2 - 4x + 2 \geqslant 0$ we will get a contradiction

$$\updownarrow$$

$$(x-2)\cdot(x-2) \geqslant 2$$

true for $x \geqslant 4$

We have assumed $x > 5$

$\square$

conclusion

For $n \in \mathbb{Z}$, $n \geqslant 5$, $2^n > n^2$

FIBONACCI NUMBERS

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$F_5 = 8$

$F_6 = 13$

etc

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

suppose smallest counterexample is $n = x$

↳ $F_x > 1.7^x$

we want a contradiction, so most likely this will involve F_{x-1}

it will be hard to use only F_x & F_{x-1}

so why not use F_{x-2} also: assume $x \geq 2$

↳ is $F_0 \leq 1.7^0$? yes. Is $F_1 \leq 1.7^1$? yes. OK!

$F_0 = F_1 = 1$ // for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x$ $(x \geq 2)$ & we can safely assume $F_y \leq 1.7^y$ for $y < x$

we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

$$= 1.7^{x-2} \cdot (1.7 + 1)$$

$$= 1.7^{x-2} \cdot 2.7$$

$$< 1.7^{x-2} \cdot (1.7)^2 \quad [1.7^2 = 2.89]$$

$$= 1.7^x$$

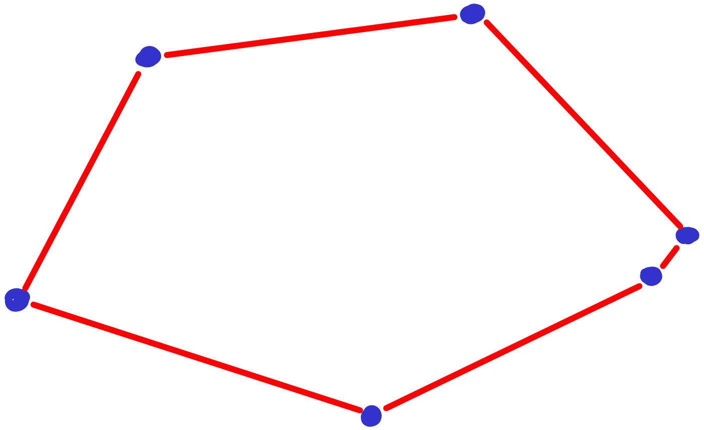
so $F_x < 1.7^x$

CONTRADICTION

$\square$

6 points in convex position.

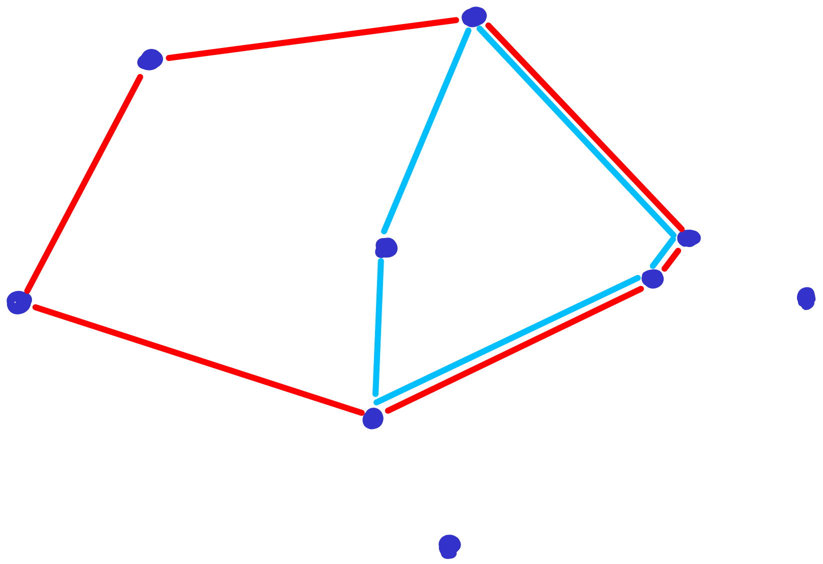
i.e., forming a hexagon



This is in 2D, aka "the plane".

x,y coordinates are real numbers, so our point set is in $\mathbb{R}^2$

Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.



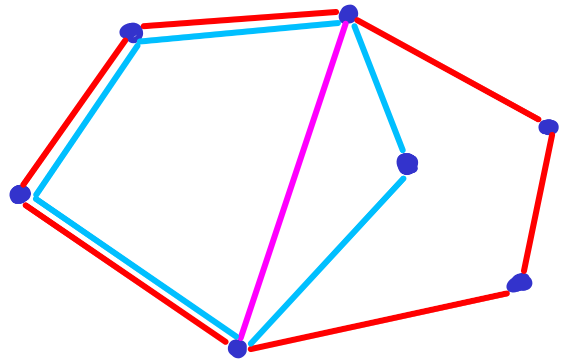
Stronger claim:

Every hexagon contains an empty pentagon

Claim: Every hexagon H contains an empty pentagon

Trivial examples:

- if H is empty, DONE.
- if H contains exactly 1 point, "split" H and then we are DONE.



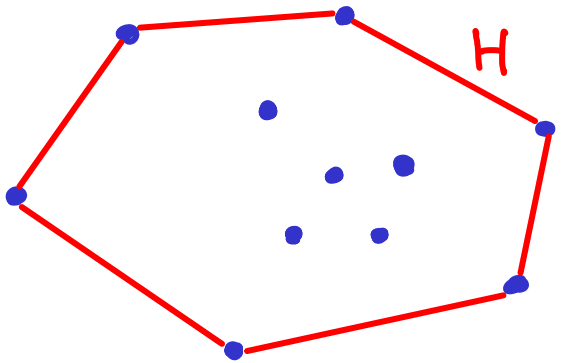
- We can order hexagons by # points inside.
- If claim is false there must be a smallest counterexample

Claim: Every hexagon H contains an empty pentagon

Proof by smallest counterexample

Choose a hexagon H containing min # pts, for which claim is false.

- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.



hypothetical smallest counterexample

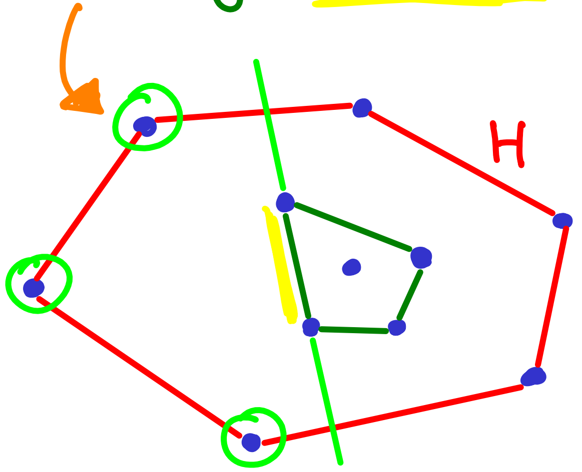
Claim: Every hexagon H contains an empty pentagon

Proof by smallest counterexample

Choose a hexagon H containing min #pts, for which claim is false.

- shown: if H contains ≤1 points, DONE → so assume ≥2 pts inside.

- if any "extreme segment" of interior points "isolates" 3 points of H ...



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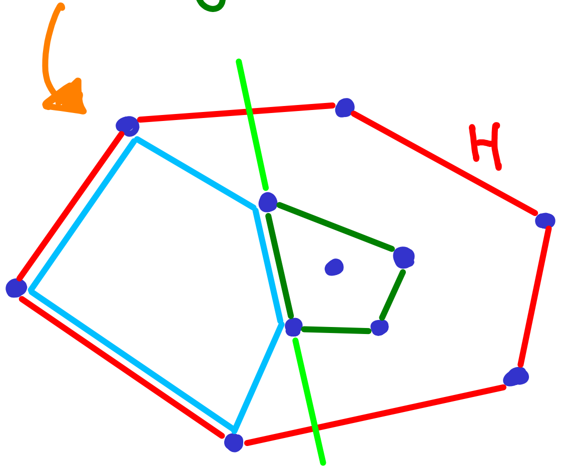
Choose a hexagon H containing min # pts, for which claim is false.

- shown: if H contains ≤ 1 points, DONE → so assume ≥ 2 pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.



this wasn't a counterexample,

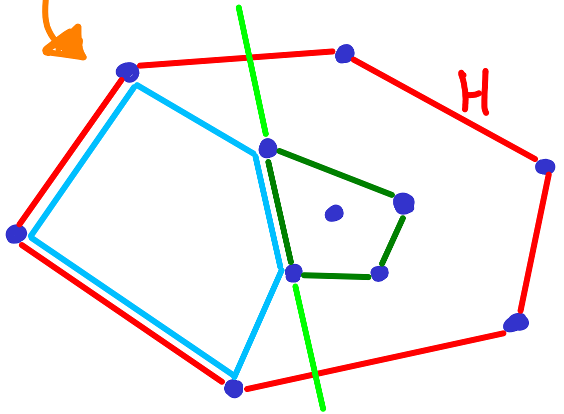
CONTRADICTION

Claim: Every hexagon H contains an empty pentagon

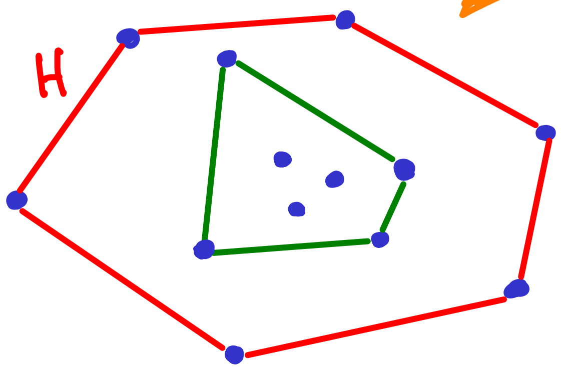
Proof by smallest counterexample

Choose a hexagon H containing min #pts, for which claim is false.

- <u>shown</u>: if H contains ≤ 1 points, DONE → so assume ≥ 2 pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.
  ↳ so every such segment isolates 1 or 2 points.



{invalid H

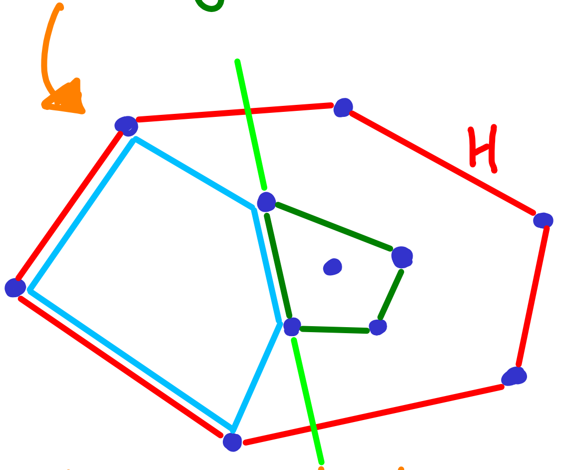


Claim: Every hexagon H contains an empty pentagon

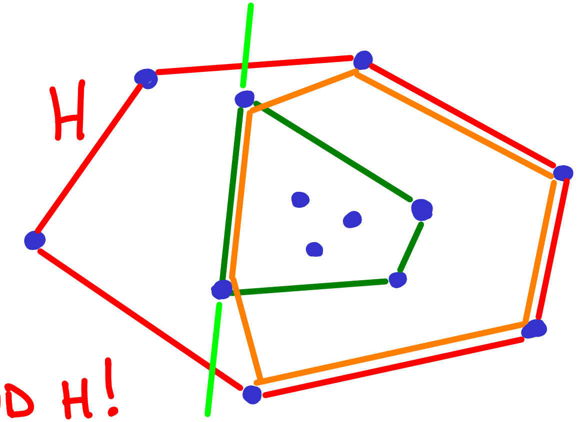
Proof by smallest counterexample

Choose a hexagon H containing min #pts, for which claim is false.

- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.
  - ↳ so every such segment isolates 1 or 2 points.



↳ use one segment & form a hexagon H' containing fewer points than H.



If H is smallest counterexample, claim is true for H' AND H!

□

proof by INDUCTION

like proof by smallest counterexample,

(1) prove your claim for a base case (should be ~easy)

like proof by smallest counterexample,

(2) focus on two "neighboring" cases [call them n-1 & n]

"unlike" proof by smallest counterexample, ... which proves $(A \wedge \neg B) = F$

which is the same

(3) show that if the claim is true for n-1
then it is true for n

$A \to B$