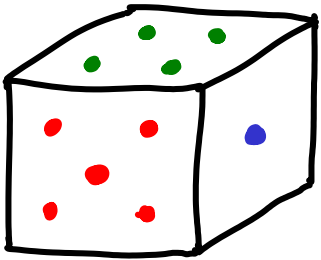


DISCRETE PROBABILITY

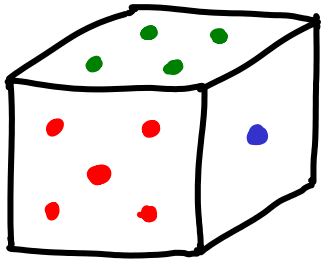
DISCRETE PROBABILITY

Roll a die ... Possible outcomes : $\{1, 2, 3, 4, 5, 6\}$



DISCRETE PROBABILITY

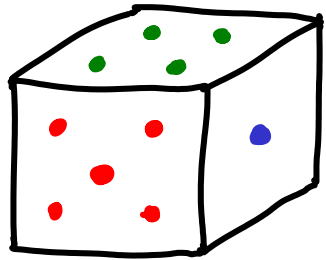
Roll a die ... Possible outcomes: $\{1, 2, 3, 4, 5, 6\}$



Each has a probability: $\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$

DISCRETE PROBABILITY

Roll a die ...



Possible outcomes :

sample space

$\{1, 2, 3, 4, 5, 6\}$

Each has a probability:

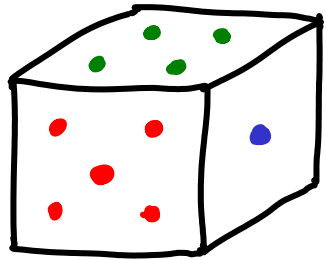
$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$

sum to 1

DISCRETE PROBABILITY

Roll a die ... Possible outcomes: $\{1, 2, 3, 4, 5, 6\}$

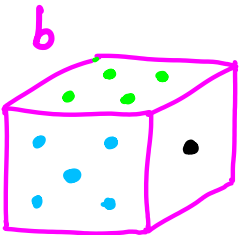
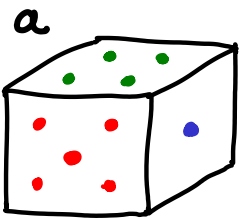
sample space



Each has a probability: $\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$

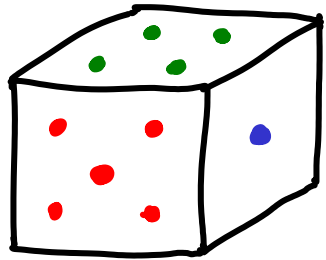
sum to 1

Roll 2 dice ... sample space $\rightarrow$?



DISCRETE PROBABILITY

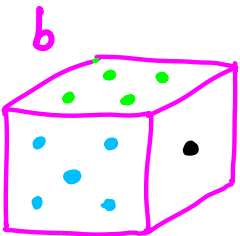
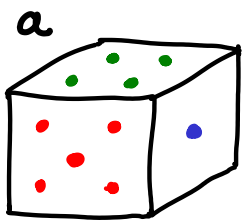
Roll a die ... Possible outcomes: $\{1, 2, 3, 4, 5, 6\}$



Each has a probability: $\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$

sum to 1

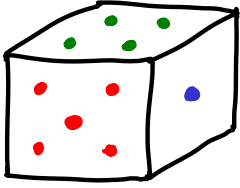
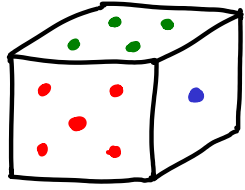
Roll 2 dice ... sample space $\rightarrow \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), \dots$



prob. $\rightarrow \frac{1}{36}$

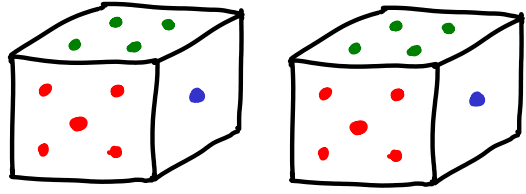
$\vdots$
 $(6,1), (6,2), \dots \dots (6,6)\}$

Roll 2 indistinguishable dice ...



sample space $\rightarrow$ $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
 $(2,2), (2,3), (2,4), (2,5), (2,6),$
 $(3,3), (3,4), (3,5), (3,6),$
 $(4,4), (4,5), (4,6),$
 $(5,5), (5,6),$
 $(6,6)\}$

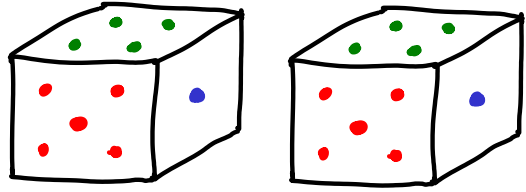
Roll 2 indistinguishable dice ...



sample space $\rightarrow$ $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
 $(2,2), (2,3), (2,4), (2,5), (2,6),$
 $(3,3), (3,4), (3,5), (3,6),$
 $(4,4), (4,5), (4,6),$
 $(5,5), (5,6),$
 $(6,6)\}$

probability of observing
some specific outcome (a,b) ?

Roll 2 indistinguishable dice ...

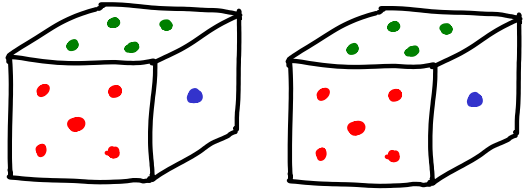


sample space $\rightarrow$ $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
 $(2,2), (2,3), (2,4), (2,5), (2,6),$
 $(3,3), (3,4), (3,5), (3,6),$
 $(4,4), (4,5), (4,6),$
 $(5,5), (5,6),$
 $(6,6)\}$

probability of observing
some specific outcome (a,b) ?

- if $a=b$, $\frac{1}{36}$

Roll 2 indistinguishable dice ...



sample space $\rightarrow$ $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
 $(2,2), (2,3), (2,4), (2,5), (2,6),$
 $(3,3), (3,4), (3,5), (3,6),$
 $(4,4), (4,5), (4,6),$
 $(5,5), (5,6),$
 $(6,6)\}$

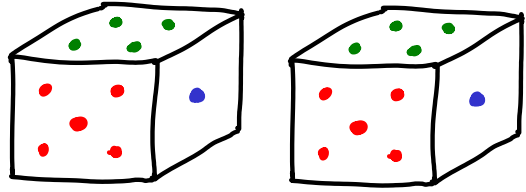
$(1,4)$ or $(4,1)$

probability of observing
some specific outcome (a,b) ?

- if $a=b$, $\frac{1}{36}$

- if $a \neq b$, $\frac{2}{36}$

Roll 2 indistinguishable dice ...



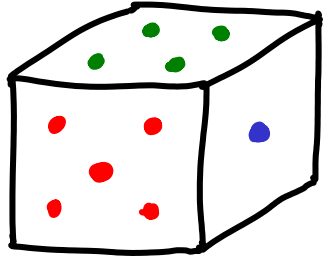
sample space $\rightarrow$ $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
 $(2,2), (2,3), (2,4), (2,5), (2,6),$
 $(3,3), (3,4), (3,5), (3,6),$
 $(4,4), (4,5), (4,6),$
 $(5,5), (5,6),$
 $(6,6)\}$

probability of observing
some specific outcome (a,b) ?

- if $a=b$, $\frac{1}{36} \rightarrow \left(6 \cdot \frac{1}{36} = \frac{6}{36} \right)$
- if $a \neq b$, $\frac{2}{36} \rightarrow \left(15 \cdot \frac{2}{36} = \frac{30}{36} \right)$

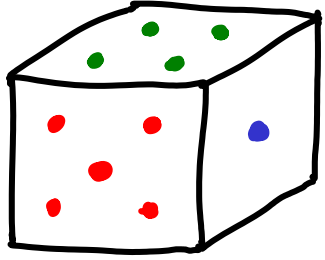
$\frac{6}{36} + \frac{30}{36} = 1$

Roll a die



$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

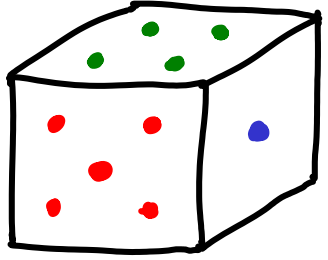
Roll a die



$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

$$P(\underbrace{\text{roll even}}_{\text{event}}) = P(\{2, 4, 6\}) = P(2) + P(4) + P(6) = \frac{1}{2}$$

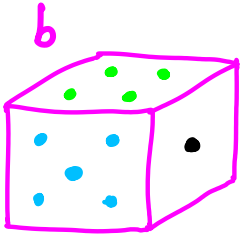
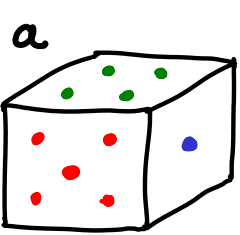
Roll a die



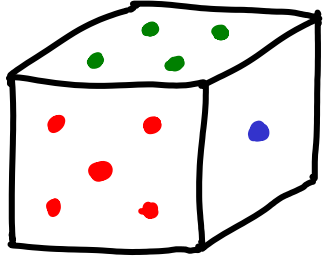
$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

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Roll 2 dice... $P(\underbrace{\text{sum} = 7}_{\text{event}}) \dots$



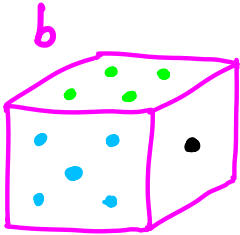
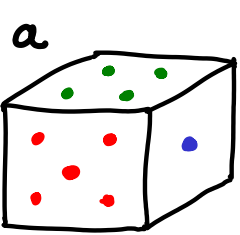
Roll a die



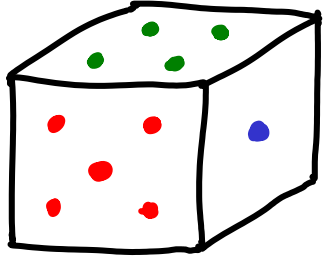
$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

$$P(\underbrace{\text{roll even}}_{\text{event}}) = P(\{2, 4, 6\}) = P(2) + P(4) + P(6) = \frac{1}{2}$$

Roll 2 dice... $P(\underbrace{\text{sum} = 7}_{\text{event}}) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\})$



Roll a die



$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

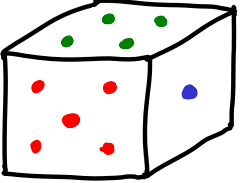
$$P(\underbrace{\text{roll even}}_{\text{event}}) = P(\{2, 4, 6\}) = P(2) + P(4) + P(6) = \frac{1}{2}$$

Roll 2 dice...

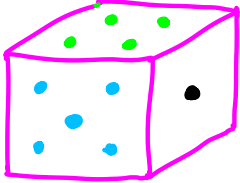
$$P(\underbrace{\text{sum} = 7}_{\text{event}}) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\})$$

$$= P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1)$$

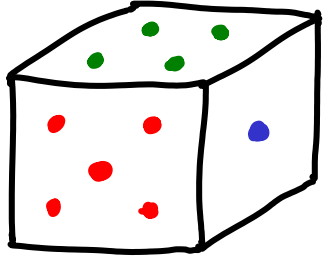
a



b



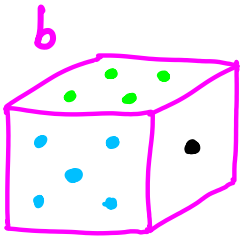
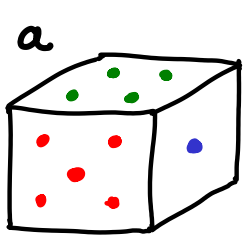
Roll a die



$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

$$P(\underbrace{\text{roll even}}_{\text{event}}) = P(\{2, 4, 6\}) = P(2) + P(4) + P(6) = \frac{1}{2}$$

Roll 2 dice...

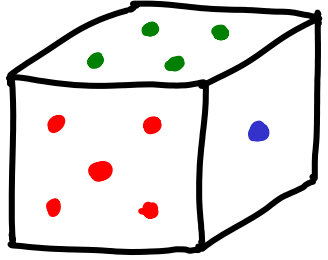


$$P(\text{sum} = 7) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\})$$

$$= P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1)$$

$$= 6 \cdot \frac{1}{36} = \frac{1}{6}$$

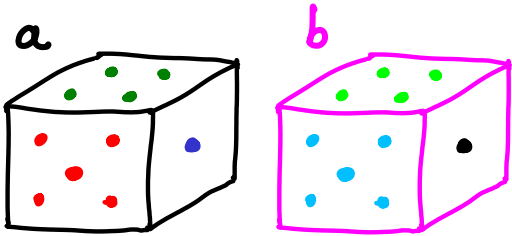
Roll a die



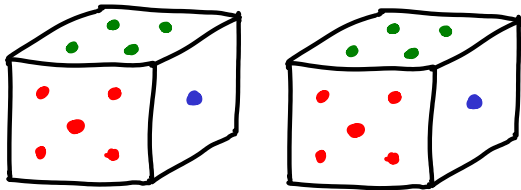
$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

$$P(\underbrace{\text{roll even}}_{\text{event}}) = P(\{2, 4, 6\}) = P(2) + P(4) + P(6) = \frac{1}{2}$$

Roll 2 dice...

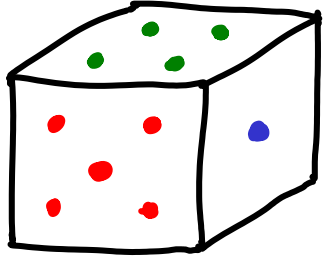


$$\begin{aligned} P(\text{sum} = 7) &= P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}) \\ &= P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1) \\ &= 6 \cdot \frac{1}{36} = \frac{1}{6} \end{aligned}$$



$$P(\text{sum} = 7) = ?$$

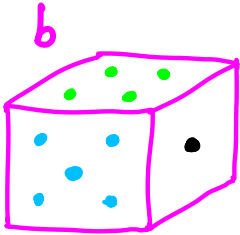
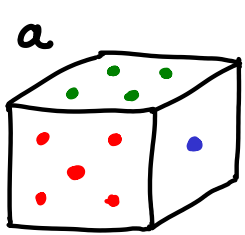
Roll a die



$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

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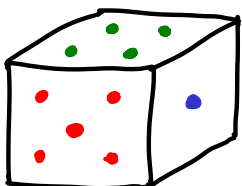
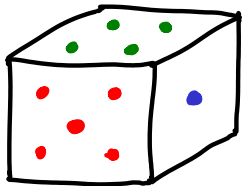
Roll 2 dice...



$$P(\underbrace{\text{sum} = 7}_{\text{event}}) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\})$$

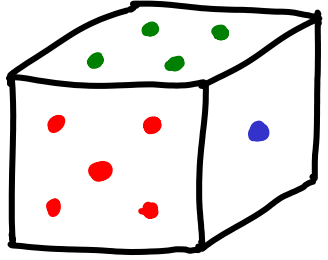
$$= P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1)$$

$$= 6 \cdot \frac{1}{36} = \frac{1}{6}$$



$$P(\text{sum} = 7) = P(\{(1,6), (2,5), (3,4)\})$$

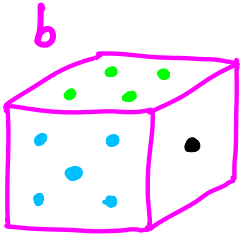
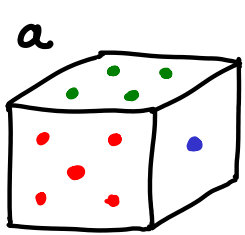
Roll a die



$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

$$P(\underbrace{\text{roll even}}_{\text{event}}) = P(\{2, 4, 6\}) = P(2) + P(4) + P(6) = \frac{1}{2}$$

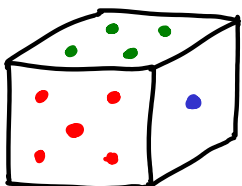
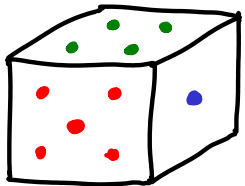
Roll 2 dice...



$$P(\underbrace{\text{sum} = 7}_{\text{event}}) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\})$$

$$= P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1)$$

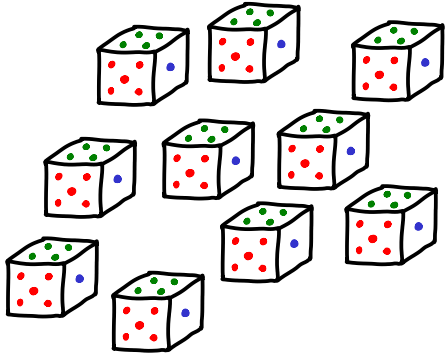
$$= 6 \cdot \frac{1}{36} = \frac{1}{6}$$



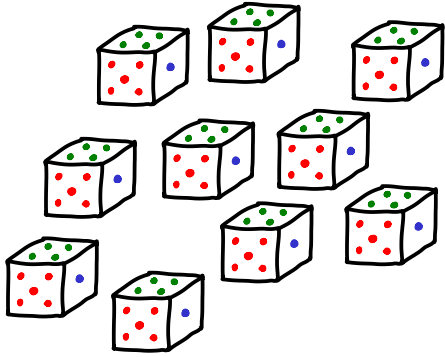
$$P(\text{sum} = 7) = P(\{(1,6), (2,5), (3,4)\}) = 3 \cdot \frac{2}{36} = \frac{1}{6}$$

Roll 10 dice. (or 1 die 10 times)

Probability we observe no 1's ?



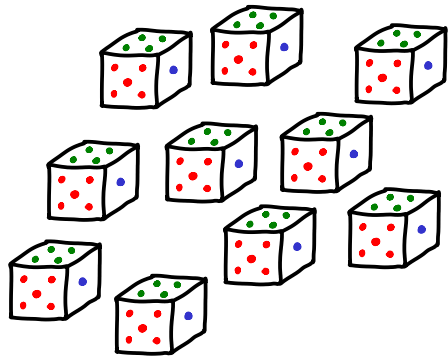
Roll 10 dice. (or 1 die 10 times) Probability we observe no 1's ?



• Sample space size:

6^{10} > 60 million
including duplicates

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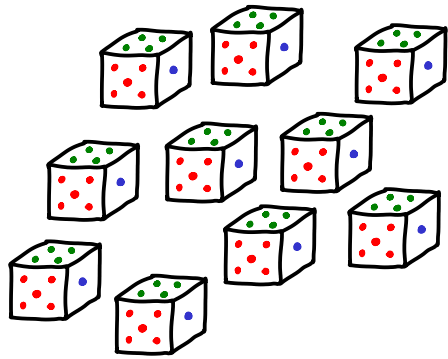


• Sample space size: 6^{10} > 60 million

including
duplicates

• How many outcomes have no 1's ?

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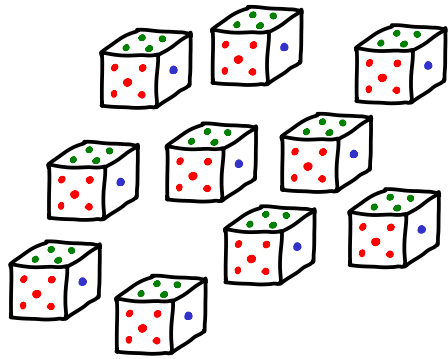


• Sample space size: 6^{10} > 60 million

including
duplicates

• How many outcomes have no 1's ? $\rightarrow 5^{10}$

Roll 10 dice. (or 1 die 10 times) Probability we observe no 1's ?

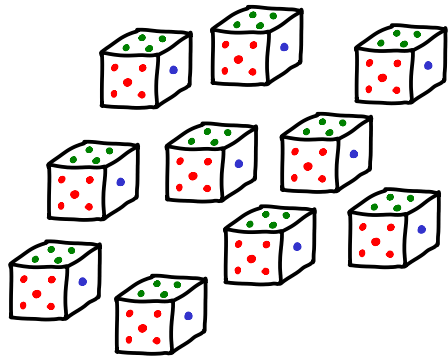


- Sample space size: 6^{10} > 60 million
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so we get $\frac{5^{10}}{6^{10}} = \left(\frac{5}{6}\right)^{10} \sim 16\%$

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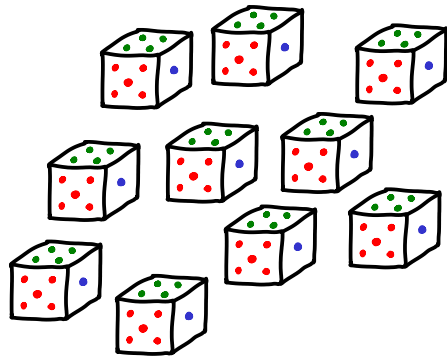
- How many outcomes have no 1's ? $\rightarrow 5^{10}$

so we get $\frac{5^{10}}{6^{10}} = \left(\frac{5}{6}\right)^{10} \sim 16\%$

OR...

- for each roll, $P(\text{no } 1) = \frac{5}{6}$

Roll 10 dice. (or 1 die 10 times) Probability we observe no 1's ?



- Sample space size: 6^{10} > 60 million
including duplicates

- How many outcomes have no 1's ? $\rightarrow 5^{10}$

so we get $\frac{5^{10}}{6^{10}} = \left(\frac{5}{6}\right)^{10} \sim 16\%$

OR...

- for each roll, $P(\text{no } 1) = \frac{5}{6}$
 - each roll/die is independent
- $\left(\frac{5}{6}\right)^{10}$

(more later)

Poker: 52 cards (13 types; 4 of each)

You get 5 cards

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- Probability 4 of 5 are the same type? "4 of a kind"

e.g., 3, 3, 3, 3, 7 OR 8, 8, 8, J, 8

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↳ order doesn't matter, no repetition

- How many "4 of a kind"?

Poker: 52 cards (13 types; 4 of each)

You get 5 cards

- Probability 4 of 5 are the same type? "4 of a kind"

e.g., 3, 3, 3, 3, 7 OR 8, 8, 8, J, 8

- How many ways can you get 5 cards? $\binom{52}{5}$ sample space

↳ order doesn't matter, no repetition

- How many "4 of a kind"?

13 types. For each, 48 remaining options as 5th card

Poker: 52 cards (13 types; 4 of each)

You get 5 cards

- Probability 4 of 5 are the same type? "4 of a kind"

e.g., 3, 3, 3, 3, 7 OR 8, 8, 8, J, 8

- How many ways can you get 5 cards? $\binom{52}{5}$ sample space

↳ order doesn't matter, no repetition

- How many "4 of a kind"? 13 · 48

13 types. For each, 48 remaining options as 5th card

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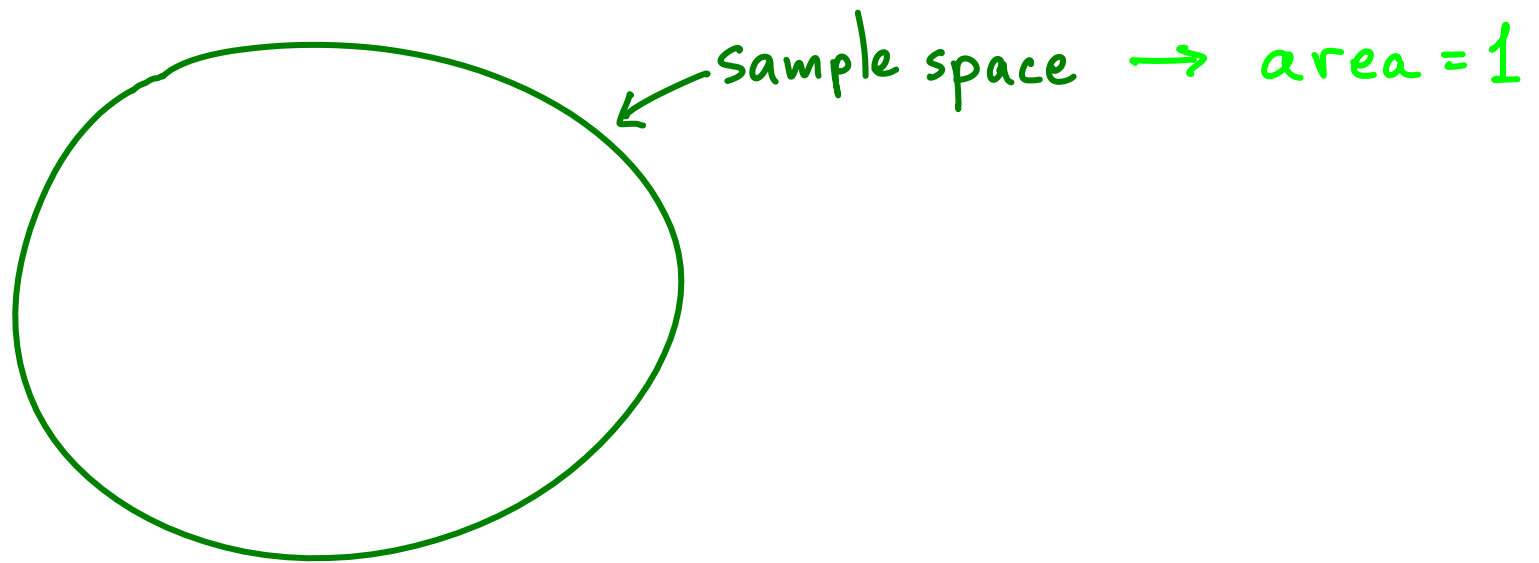
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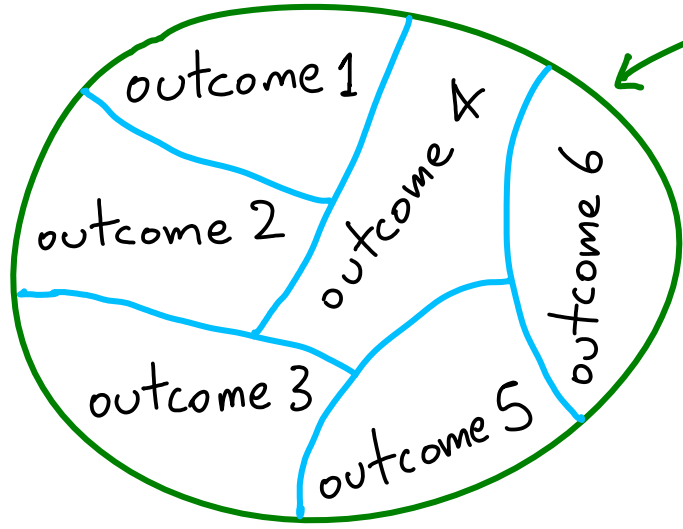
13 types. For each, 48 remaining options as 5th card

$$\rightarrow \frac{13 \cdot 48}{\binom{52}{5}} = \frac{1}{4165} \sim 0.00024 = 0.024\%$$

PROBABILITY VISUALIZATION

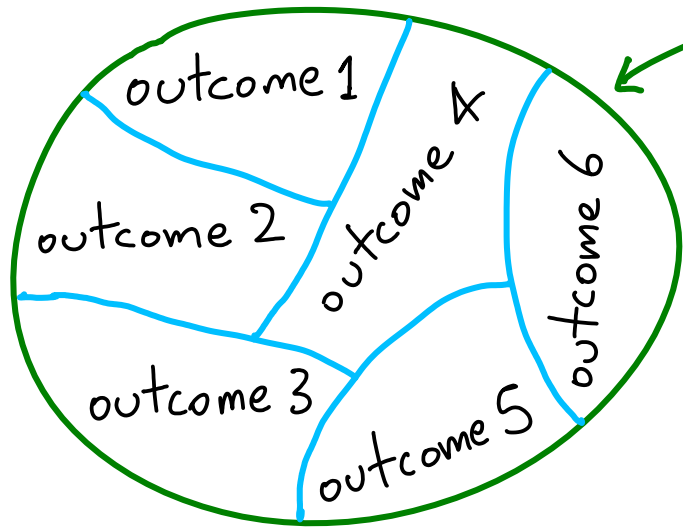


PROBABILITY VISUALIZATION



sample space $\rightarrow$ area = 1
outcomes partition sample space

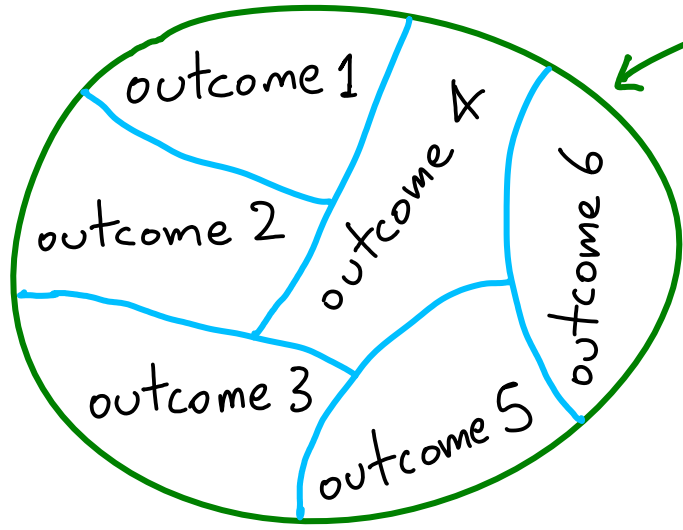
PROBABILITY VISUALIZATION



sample space $\rightarrow$ area = 1
outcomes partition sample space

e.g., roll one die $\rightarrow$ all areas equal $\frac{1}{6}$

PROBABILITY VISUALIZATION

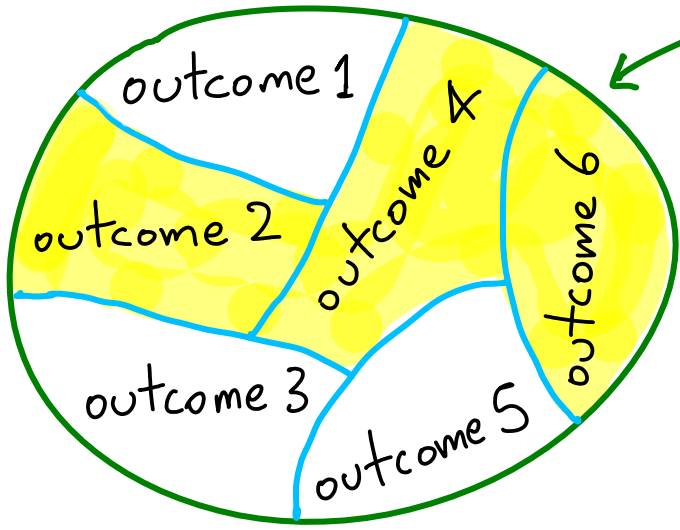


sample space $\rightarrow$ area = 1
outcomes partition sample space

e.g., roll one die $\rightarrow$ all areas equal $\frac{1}{6}$

$$P(\text{outcome } i) = \text{area}(i)$$

PROBABILITY VISUALIZATION



sample space $\rightarrow$ area = 1
outcomes partition sample space

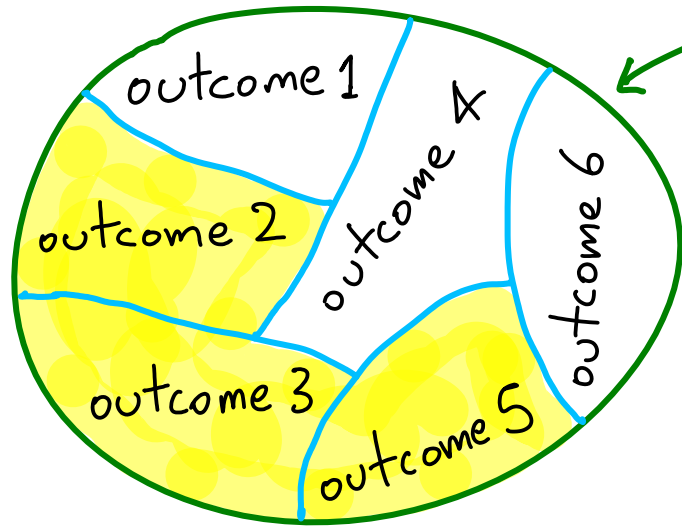
e.g., roll one die $\rightarrow$ all areas equal $\frac{1}{6}$

$$P(\text{outcome } i) = \text{area}(i)$$

$$P(\text{event}) = \text{sum of appropriate areas}$$

$\{2, 4, 6\}$
● $P(\text{even}) = \frac{3}{6}$

PROBABILITY VISUALIZATION



sample space $\rightarrow$ area = 1
outcomes partition sample space

e.g., roll one die $\rightarrow$ all areas equal $\frac{1}{6}$

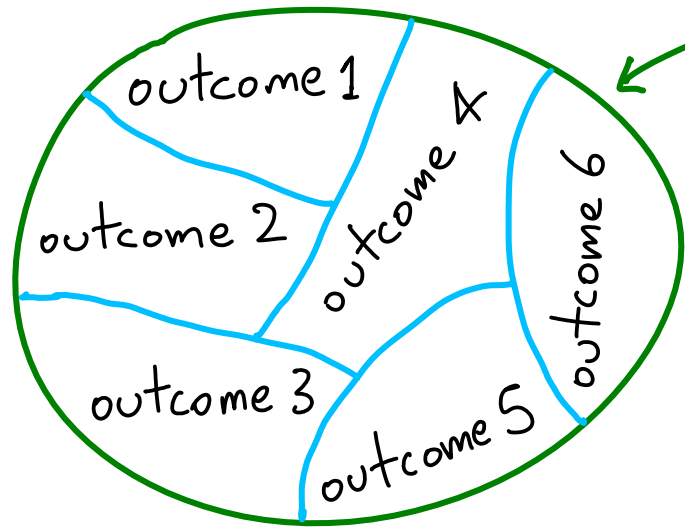
$$P(\text{outcome } i) = \text{area}(i)$$

$P(\text{event}) = \text{sum of appropriate areas}$

● $\underbrace{2, 3, 5}_{\text{prime}}$
 $P(\text{prime}) = \frac{3}{6}$

$\underbrace{2, 4, 6}_{\text{even}}$
 $P(\text{even}) = \frac{3}{6}$

PROBABILITY VISUALIZATION



sample space $\rightarrow$ area = 1
outcomes partition sample space

e.g., roll one die $\rightarrow$ all areas equal $\frac{1}{6}$

$$P(\text{outcome } i) = \text{area}(i)$$

$P(\text{event}) = \text{sum of appropriate areas}$

e.g. $P(\text{prime OR even}) ?$

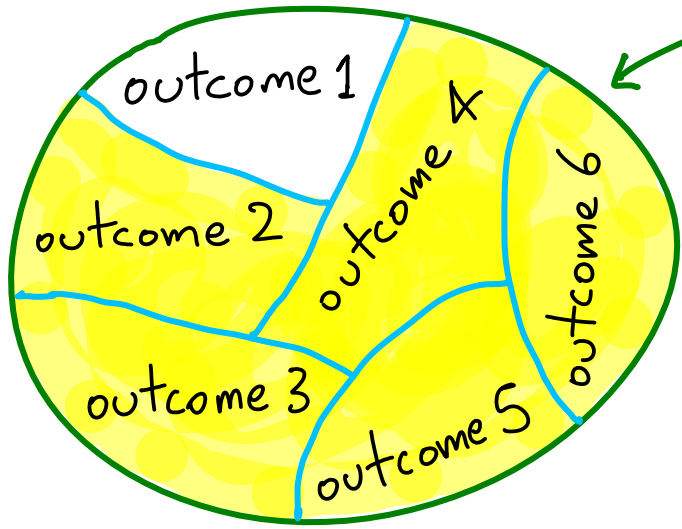
$\{2, 3, 5\}$

$$P(\text{prime}) = \frac{3}{6}$$

$\{2, 4, 6\}$

$$P(\text{even}) = \frac{3}{6}$$

PROBABILITY VISUALIZATION



sample space $\rightarrow$ area = 1
outcomes partition sample space

e.g., roll one die $\rightarrow$ all areas equal $\frac{1}{6}$

$$P(\text{outcome } i) = \text{area}(i)$$

$P(\text{event}) = \text{sum of appropriate areas}$

e.g. $P(\text{prime OR even})$

$\underbrace{2, 3, 5}$

$$P(\text{prime}) = \frac{3}{6}$$

$\underbrace{2, 4, 6}$

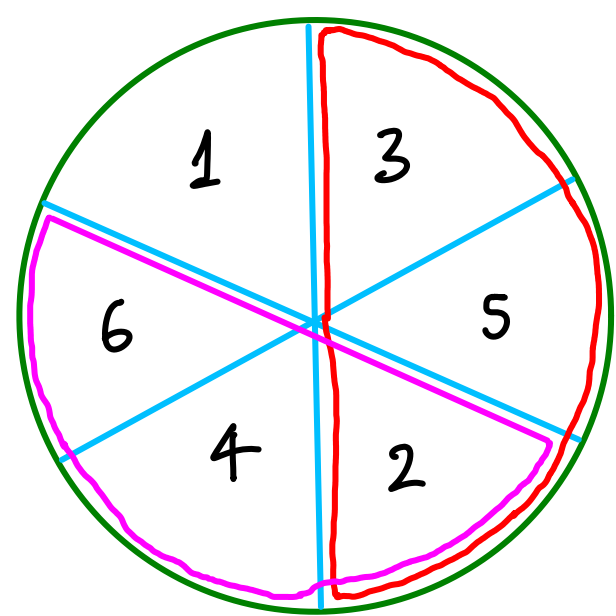
$$P(\text{even}) = \frac{3}{6}$$

$\left. \begin{array}{l} \underbrace{2, 3, 5} \\ \underbrace{2, 4, 6} \end{array} \right\} \frac{5}{6}$

avoid doublecounting

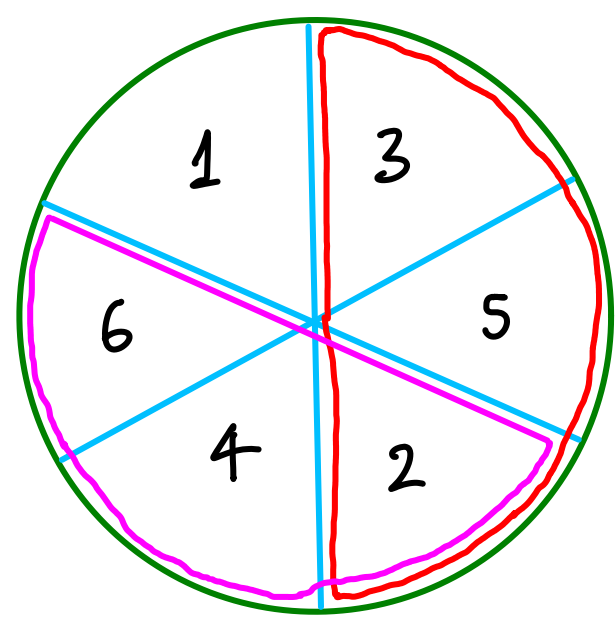
$$\text{NOT } \frac{3}{6} + \frac{3}{6}$$

areas of events can overlap



$$P(\text{prime}) = P(A)$$

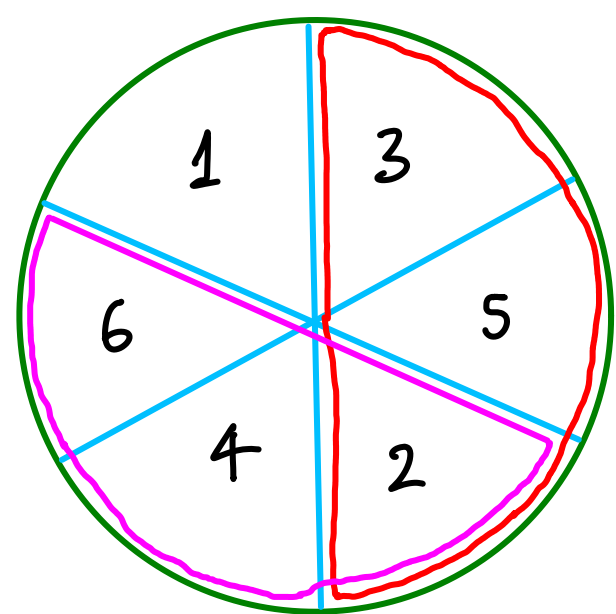
$$P(\text{even}) = P(B)$$



$$P(\text{prime}) = P(A)$$

$$P(\text{even}) = P(B)$$

$$P(\text{prime OR even}) = \underline{P(A \cup B)}$$

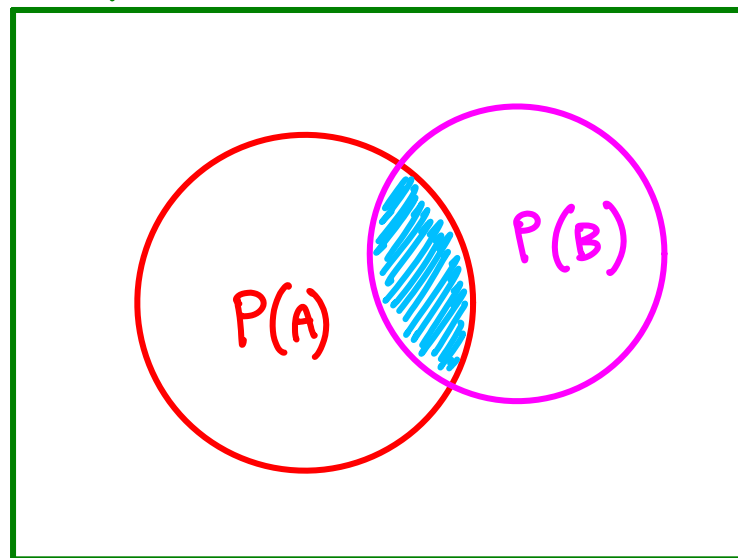


$$P(\text{prime}) = P(A)$$

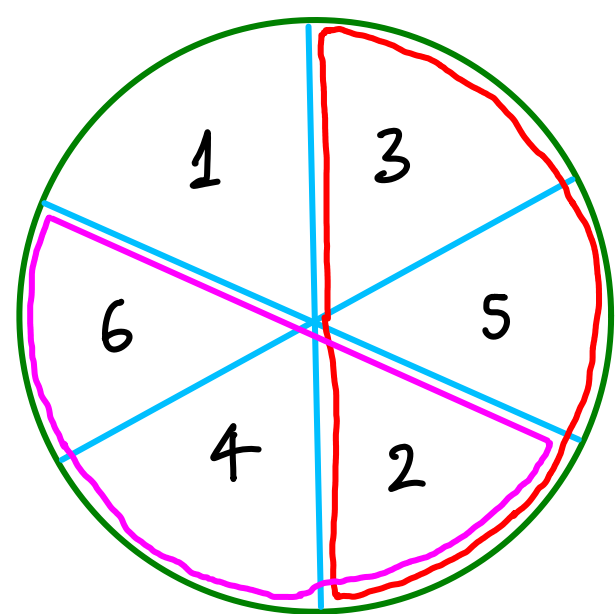
$$P(\text{even}) = P(B)$$

$$P(\text{prime OR even}) = P(A \cup B)$$

all probability space



areas not considered

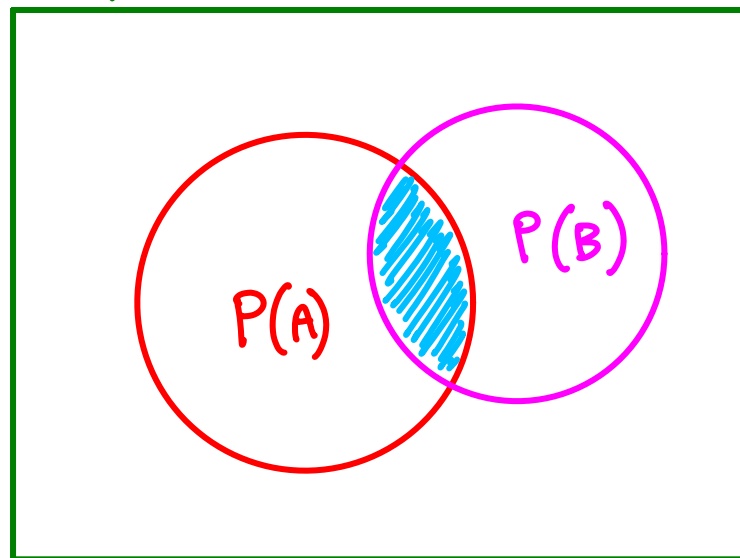


$$P(\text{prime}) = P(A)$$

$$P(\text{even}) = P(B)$$

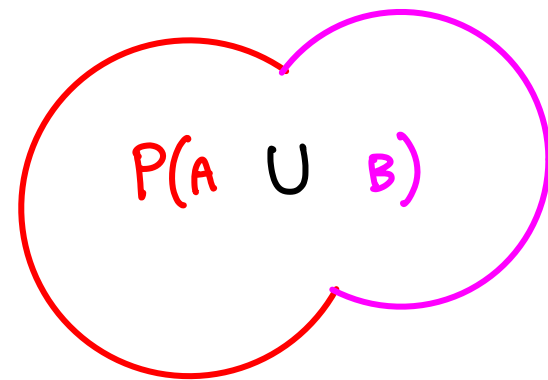
$$P(\text{prime OR even}) = P(A \cup B)$$

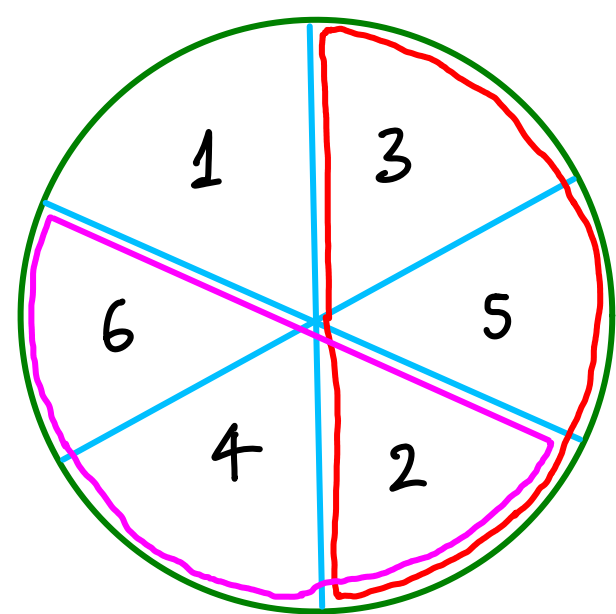
all probability space



areas not considered

"OR" :



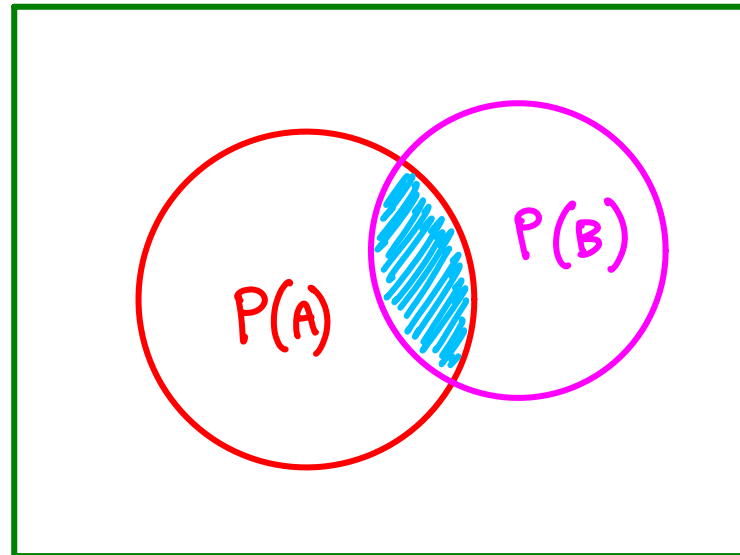


$$P(\text{prime}) = P(A)$$

$$P(\text{even}) = P(B)$$

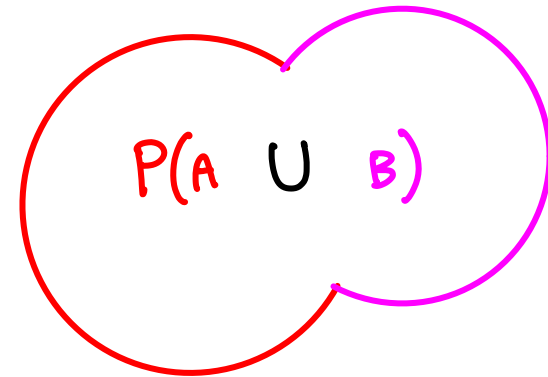
$$P(\text{prime OR even}) = P(A \cup B)$$

all probability space

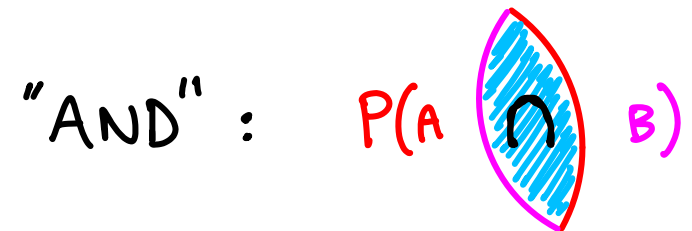


areas not considered

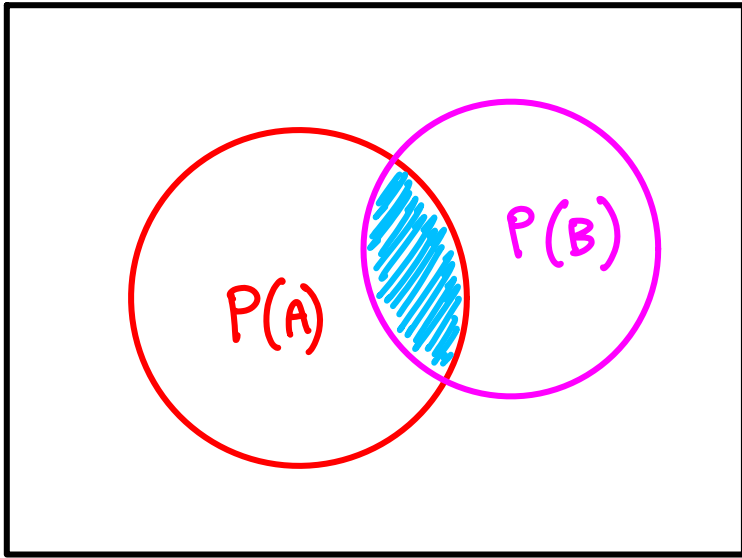
"OR" :



"AND" :

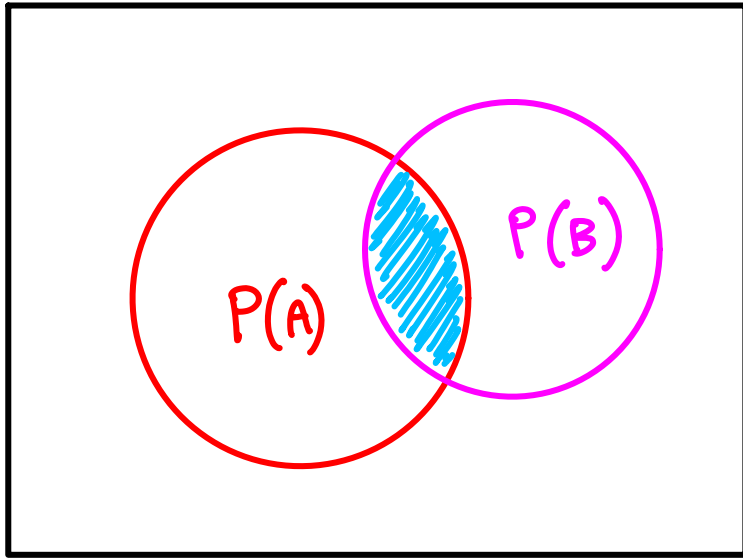


all probability space



$P(A)$ $P(B)$ ^{vs} $P(A \cup B)$ $P(A \cap B)$
?

all probability space

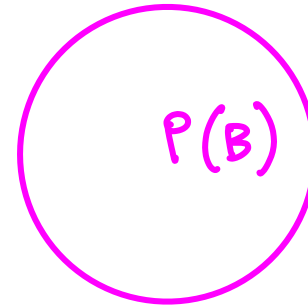
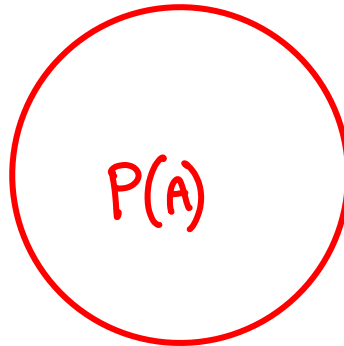
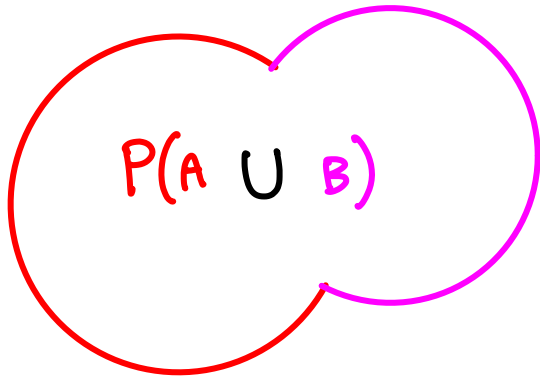


$P(A)$

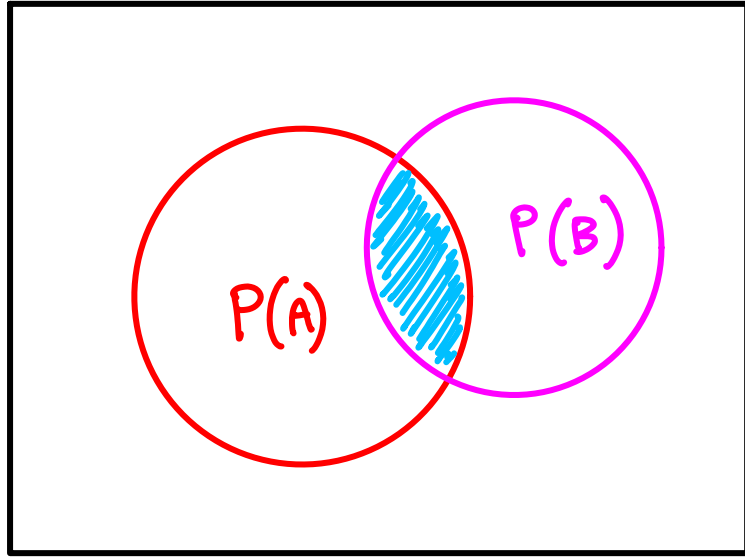
$P(B)$

$P(A \cup B)$

$P(A \cap B)$



all probability space



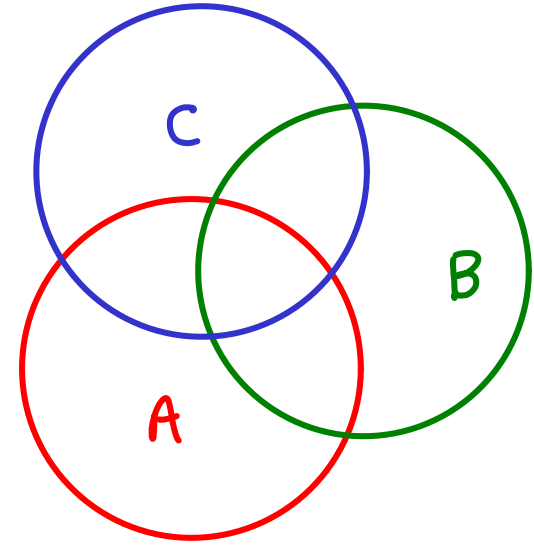
$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

INCLUSION-EXCLUSION principle

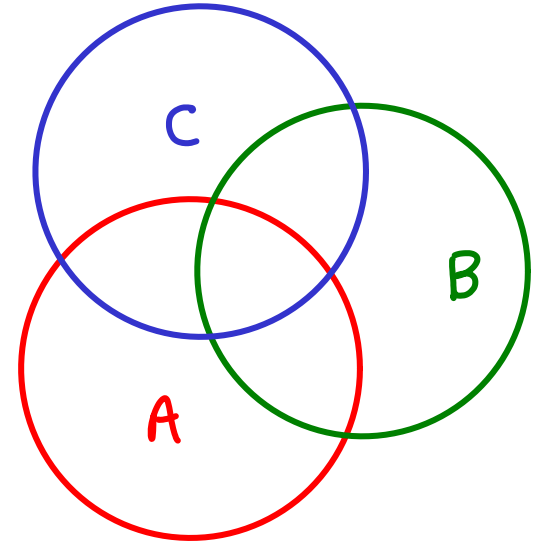
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

avoid doublecounting

General INCLUSION-EXCLUSION principle for sets

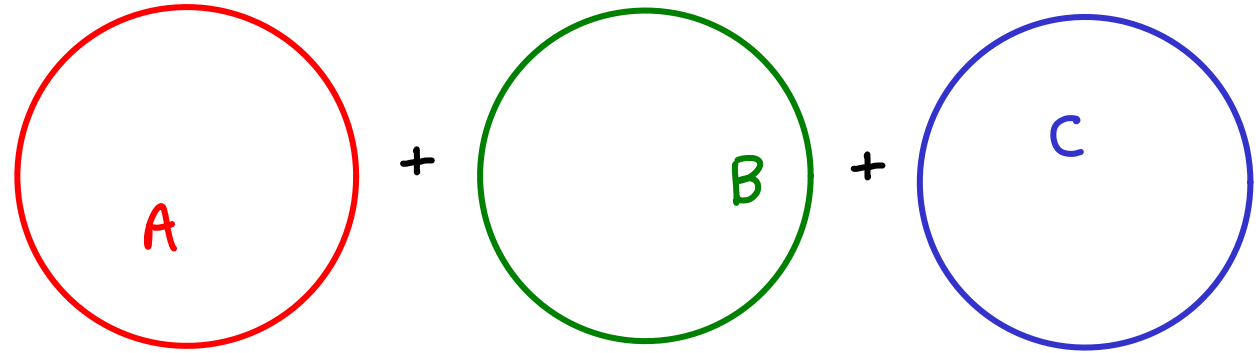
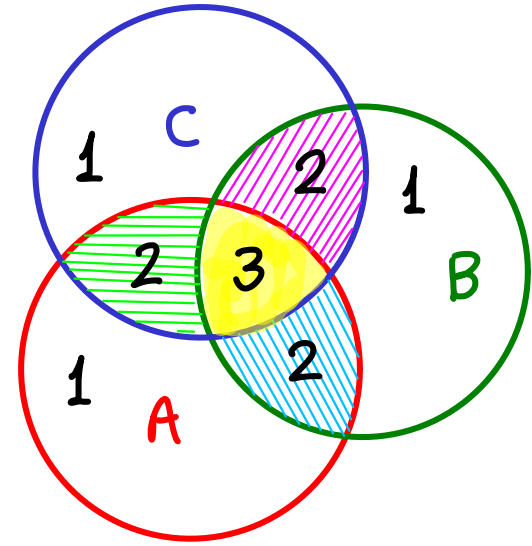


General INCLUSION-EXCLUSION principle for sets



$$|A \cup B \cup C| = ?$$

General INCLUSION-EXCLUSION principle for sets



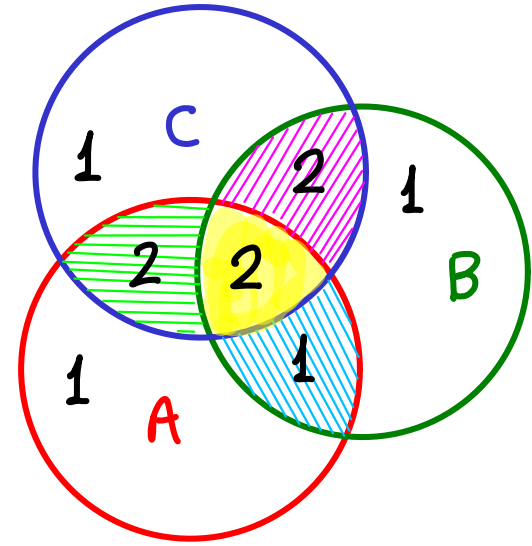
$$|A \cup B \cup C| =$$

$$= |A| + |B| + |C|$$

⋮

not done, must fix overcounting

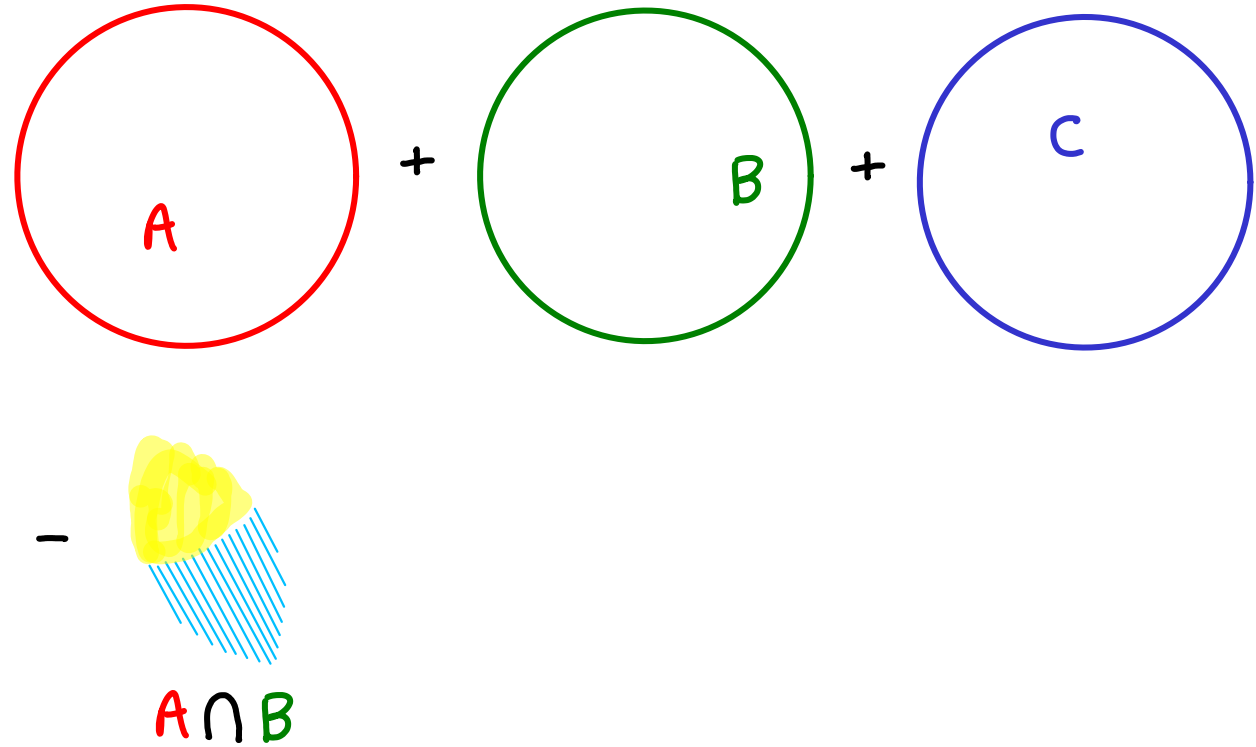
General INCLUSION-EXCLUSION principle for sets



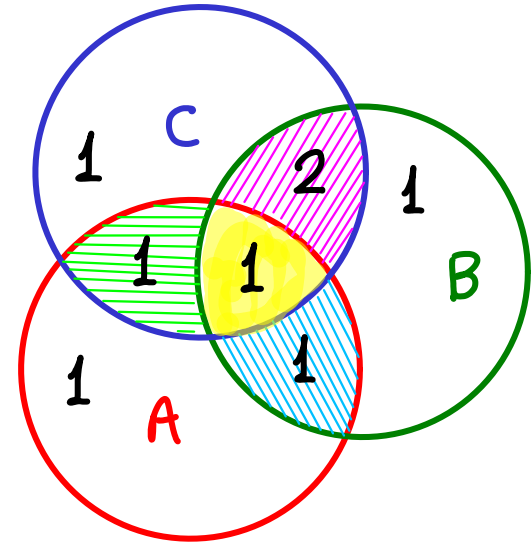
$$|A \cup B \cup C| =$$

$$= |A| + |B| + |C|$$

$$- |A \cap B| \dots$$



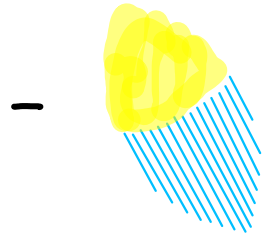
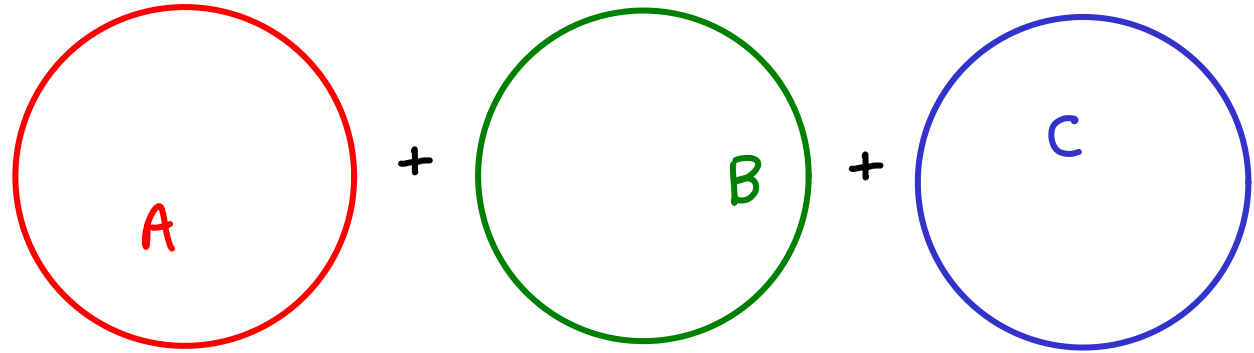
General INCLUSION-EXCLUSION principle for sets



$$|A \cup B \cup C| =$$

$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| \dots$$

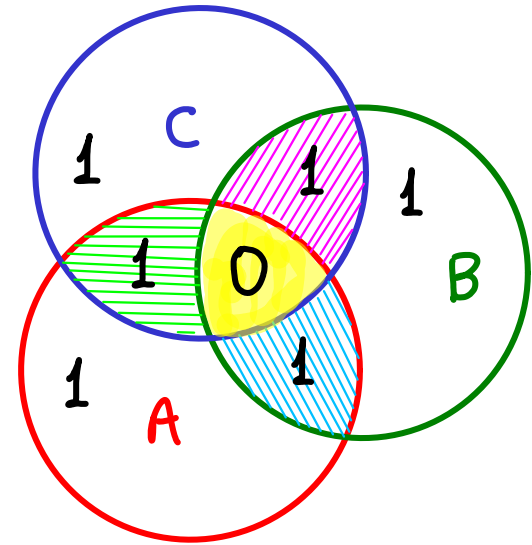


$$A \cap B$$



$$A \cap C$$

General INCLUSION-EXCLUSION principle for sets

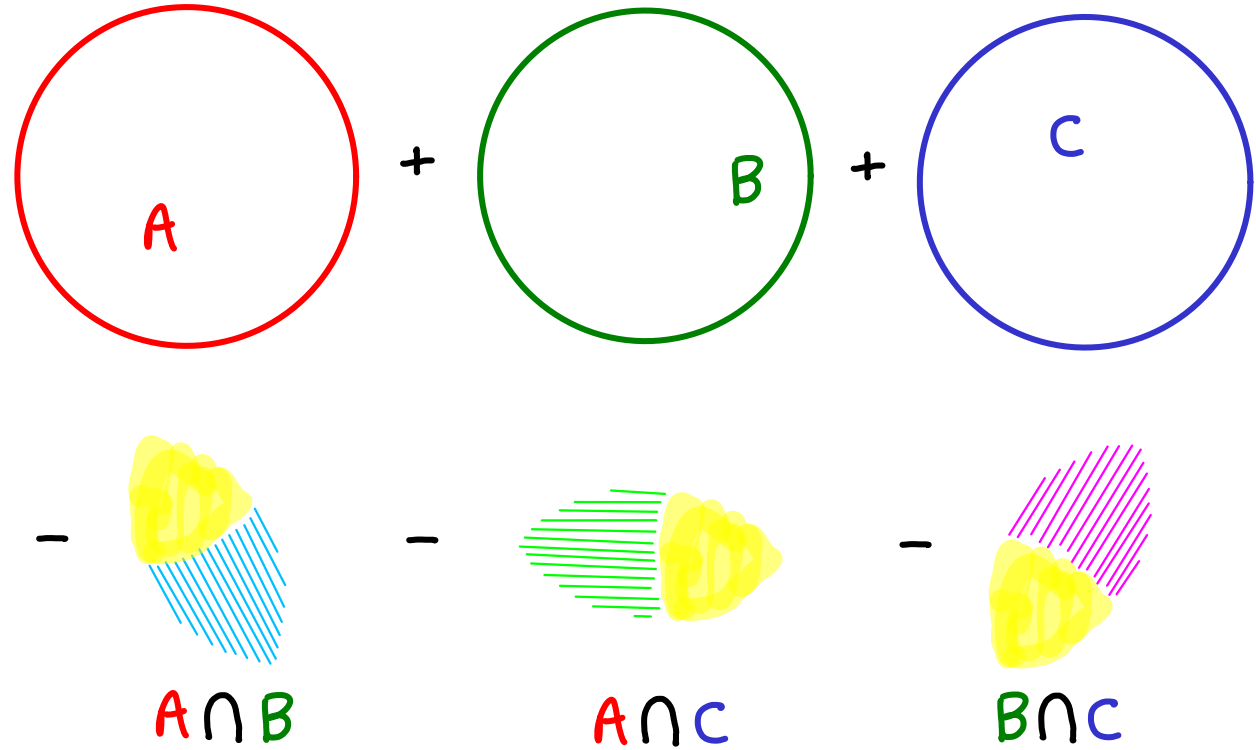


$$|A \cup B \cup C| =$$

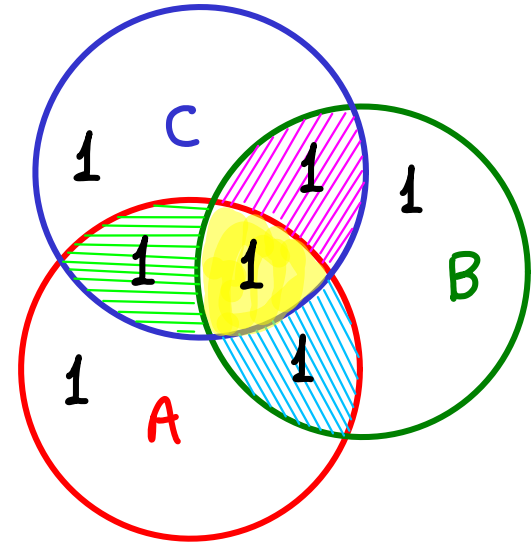
$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

...



General INCLUSION-EXCLUSION principle for sets

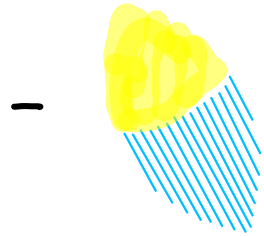
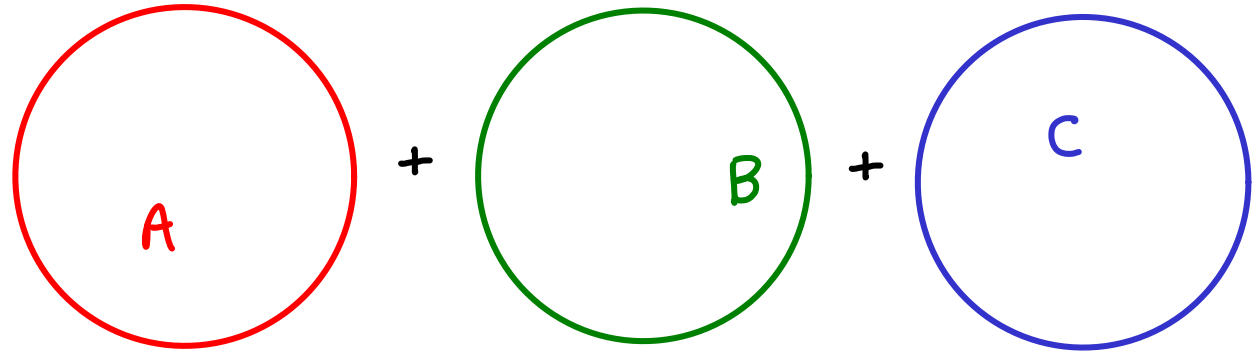


$$|A \cup B \cup C| =$$

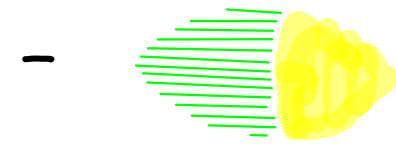
$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

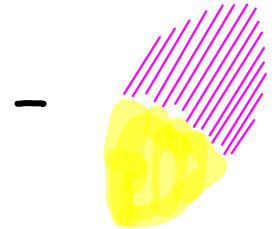
$$+ |A \cap B \cap C|$$



$$A \cap B$$



$$A \cap C$$

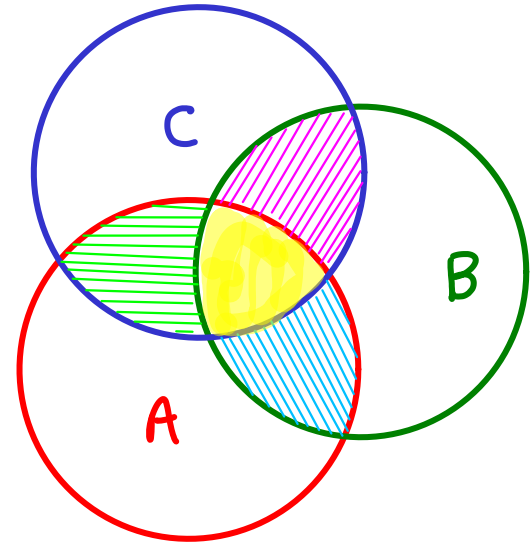


$$B \cap C$$



$$A \cap B \cap C$$

General INCLUSION-EXCLUSION principle for sets

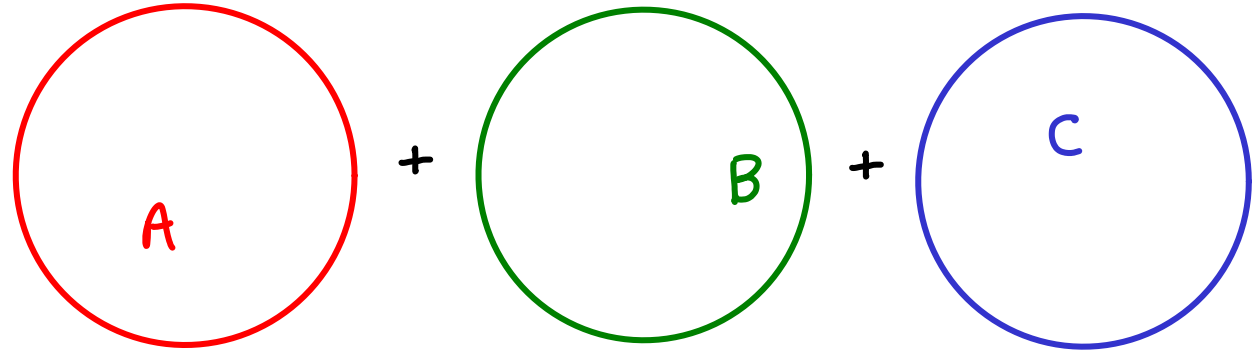


$$|A \cup B \cup C| =$$

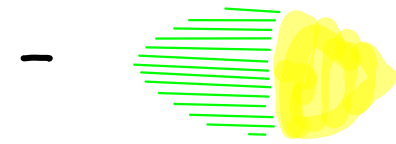
$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

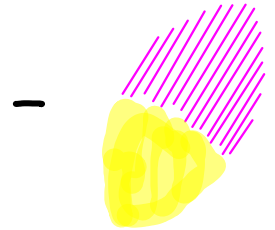
$$+ |A \cap B \cap C|$$



$$A \cap B$$



$$A \cap C$$



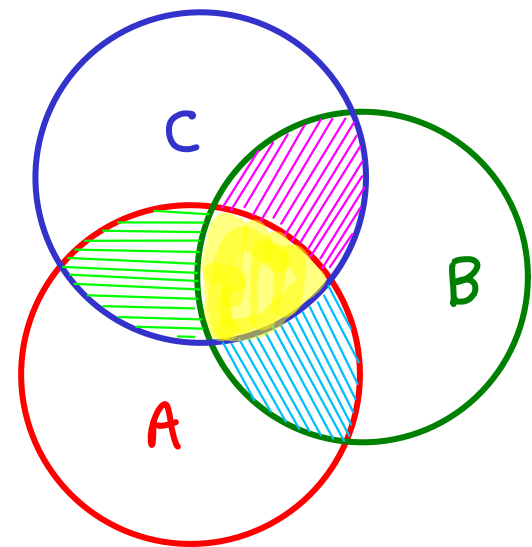
$$B \cap C$$



$$A \cap B \cap C$$

General INCLUSION-EXCLUSION principle for sets

E_i : set i



$$\left| \bigcup_{i=1}^n E_i \right| = ?$$

$$|A \cup B \cup C| =$$

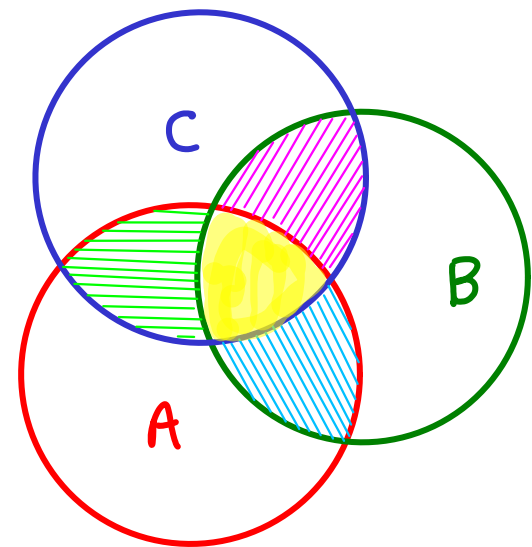
$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$

General INCLUSION-EXCLUSION principle for sets

E_i : set i



$$\left| \bigcup_{i=1}^n E_i \right| = \sum_{i=1}^n |E_i|$$

⋮

$$|A \cup B \cup C| =$$

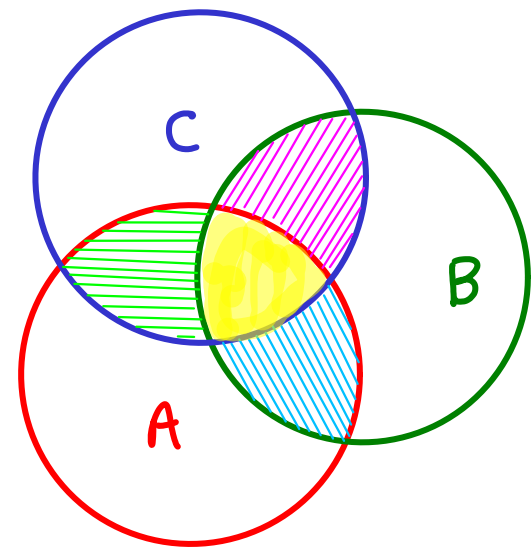
$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$

General INCLUSION-EXCLUSION principle for sets

E_i : set i



$$|A \cup B \cup C| =$$

$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

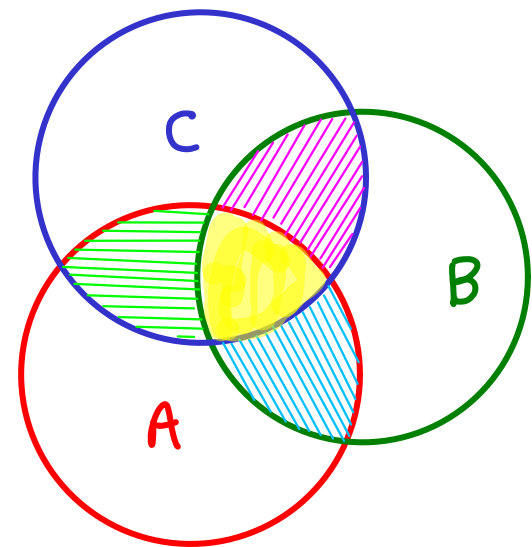
$$+ |A \cap B \cap C|$$

all pairs →

$$\left| \bigcup_{i=1}^n E_i \right| = + \sum_{i=1}^n |E_i|$$
$$- \sum_{\substack{1 \leq i < j \leq n}} |E_i \cap E_j|$$
$$\vdots$$

General INCLUSION-EXCLUSION principle for sets

E_i : set i



$$|A \cup B \cup C| =$$

$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$

all pairs $\rightarrow$

all triples $\rightarrow$

$$\left| \bigcup_{i=1}^n E_i \right| = + \sum_{i=1}^n |E_i|$$

$$- \sum_{1 \leq i < j \leq n} |E_i \cap E_j|$$

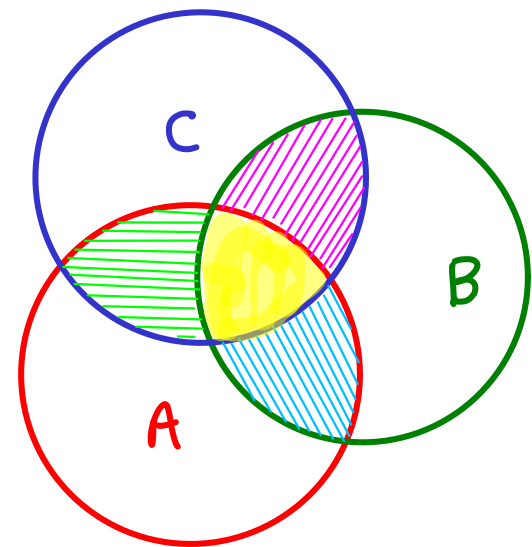
$$+ \sum_{1 \leq i < j < k \leq n} |E_i \cap E_j \cap E_k|$$

$\vdots$



General INCLUSION-EXCLUSION principle for sets

E_i : set i



$$|A \cup B \cup C| =$$

$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$

$$\left| \bigcup_{i=1}^n E_i \right| = + \sum_{i=1}^n |E_i|$$

all pairs $\rightarrow$

alternate sign

$$- \sum_{1 \leq i < j \leq n} |E_i \cap E_j|$$

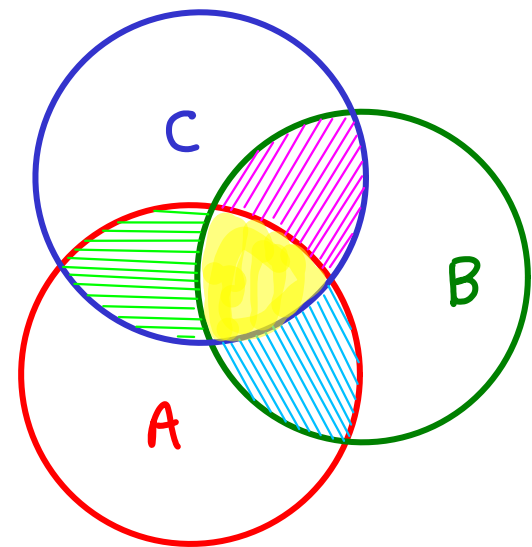
all triples $\rightarrow$

$$+ \sum_{1 \leq i < j < k \leq n} |E_i \cap E_j \cap E_k|$$

$-$
 $+$
 $-$
 $+$

General INCLUSION-EXCLUSION principle for sets

E_i : set i



$$|A \cup B \cup C| =$$

$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$

$$\left| \bigcup_{i=1}^n E_i \right| = + \sum_{i=1}^n |E_i|$$

all pairs $\rightarrow$

alternate sign

$$- \sum_{1 \leq i < j \leq n} |E_i \cap E_j|$$

all triples $\rightarrow$

$$+ \sum_{1 \leq i < j < k \leq n} |E_i \cap E_j \cap E_k|$$

(+1) if n is odd

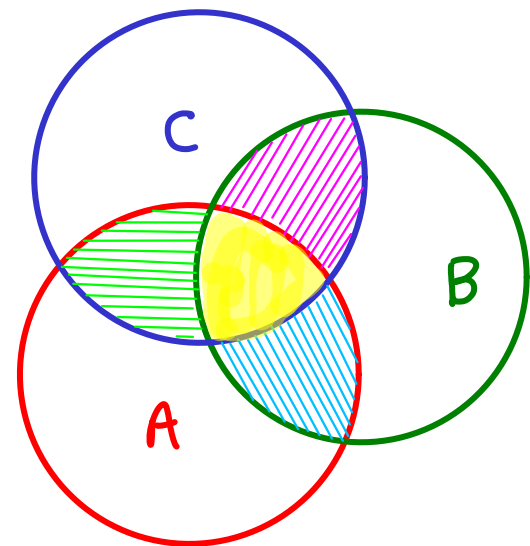
(-1) if n is even

-
+
-
+

$(-1)^{n-1} \dots$

General INCLUSION-EXCLUSION principle for sets

E_i : set i



$$|A \cup B \cup C| =$$

$$= |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$

$$\left| \bigcup_{i=1}^n E_i \right| = + \sum_{i=1}^n |E_i|$$

all pairs $\rightarrow$

alternate sign

$$- \sum_{1 \leq i < j \leq n} |E_i \cap E_j|$$

all triples $\rightarrow$

$$+ \sum_{1 \leq i < j < k \leq n} |E_i \cap E_j \cap E_k|$$

(+1) if n is odd

(-1) if n is even

"all" groups of size n

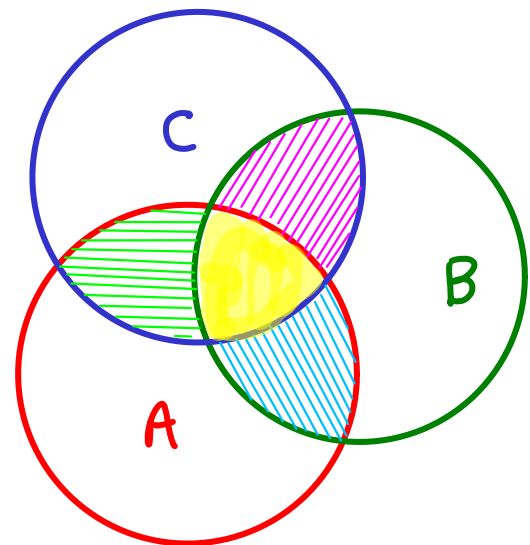
(there is only one group) $\rightarrow$

$$(-1)^{n-1} |E_1 \cap \dots \cap E_n|$$

+
-
+
-

General INCLUSION-EXCLUSION principle for sets

E_i : set i



can be proved using
Binomial theorem

$$\left| \bigcup_{i=1}^n E_i \right| = + \sum_{i=1}^n |E_i|$$

all pairs $\rightarrow$

alternate sign

$$- \sum_{1 \leq i < j \leq n} |E_i \cap E_j|$$

all triples $\rightarrow$

$$+ \sum_{1 \leq i < j < k \leq n} |E_i \cap E_j \cap E_k|$$

(+1) if n is odd

(-1) if n is even

-
+
-
+

"all" groups of size n
(there is only one group) $\rightarrow$

$$(-1)^{n-1} |E_1 \cap \dots \cap E_n|$$

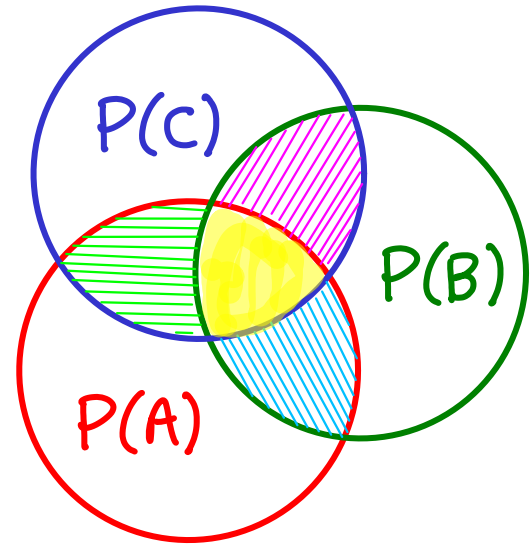
$$|A \cup B \cup C| =$$

$$= |A| + |B| + |C|$$

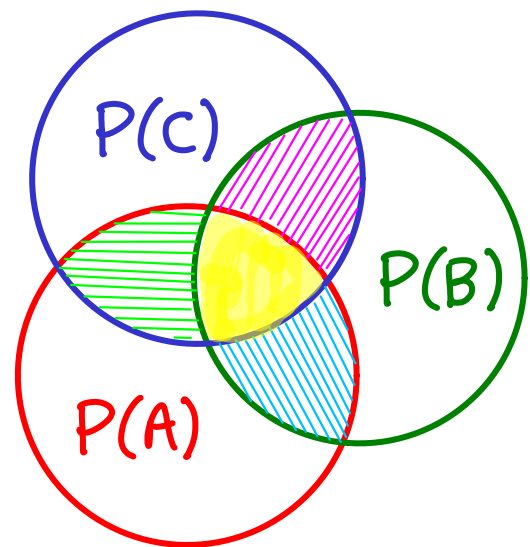
$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$

General INCLUSION-EXCLUSION principle for probability



General INCLUSION-EXCLUSION principle for probability



$$P(A \cup B \cup C) =$$

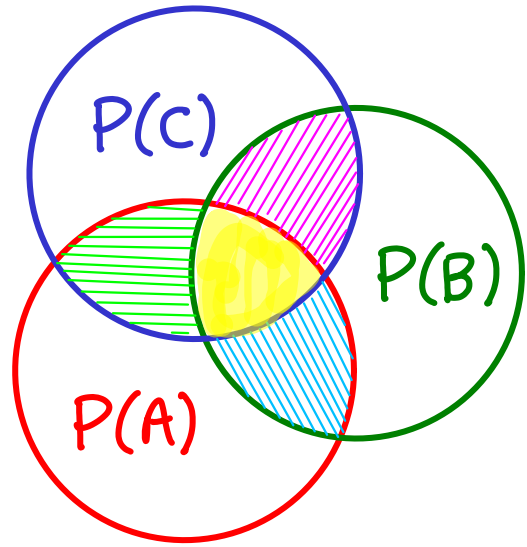
$$= P(A) + P(B) + P(C)$$

$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$

$$+ P(A \cap B \cap C)$$

General INCLUSION-EXCLUSION principle for probability

E_i : event i



$$P(A \cup B \cup C) =$$

$$= P(A) + P(B) + P(C)$$

$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$

$$+ P(A \cap B \cap C)$$

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i)$$

$$- \sum_{1 \leq i < j \leq n} P(E_i \cap E_j)$$

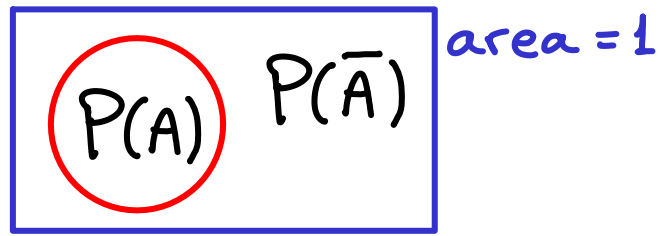
$$+ \sum_{1 \leq i < j < k \leq n} P(E_i \cap E_j \cap E_k)$$

$\vdots$

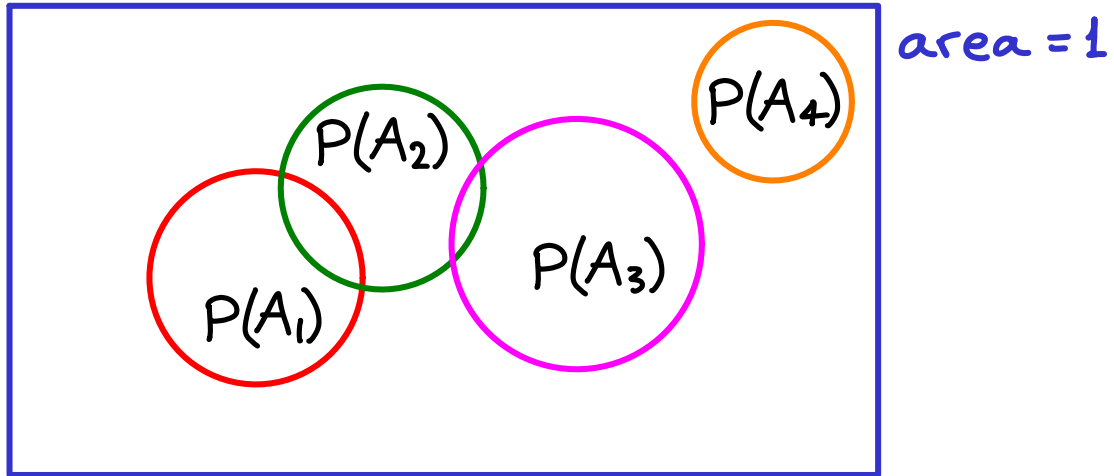
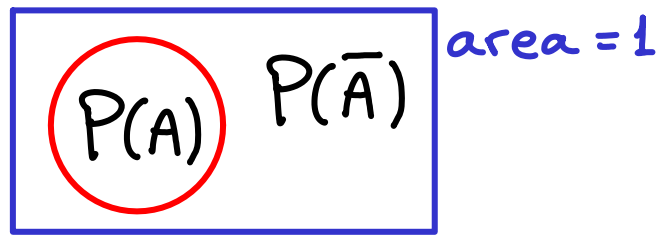
$$(-1)^{n-1} P\left(\bigcap_{i=1}^n E_i\right)$$

$P(A)$ $P(\bar{A})$ area = 1

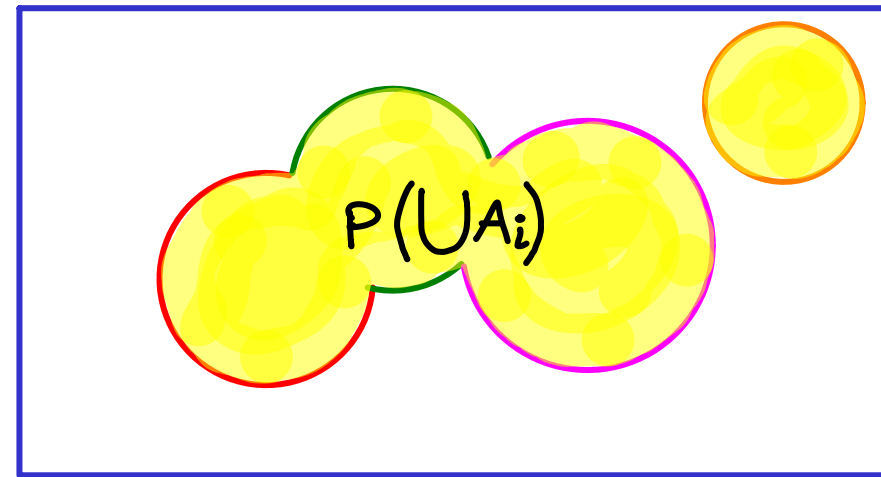
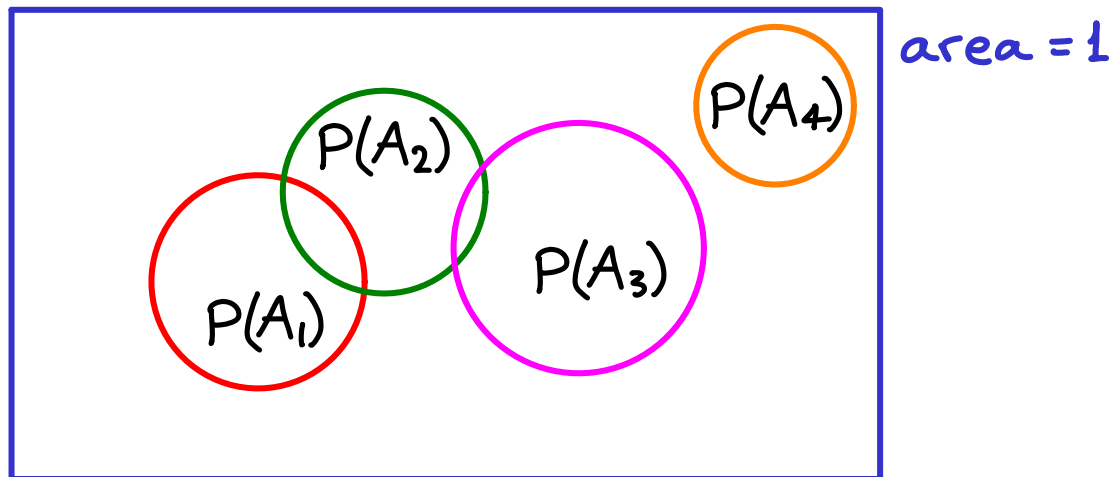
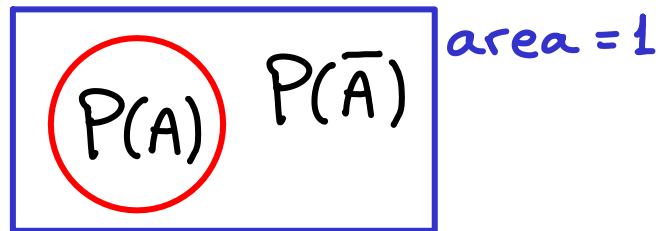
$$P(A) + P(\bar{A}) = 1$$



$$P(A) + P(\bar{A}) = 1$$

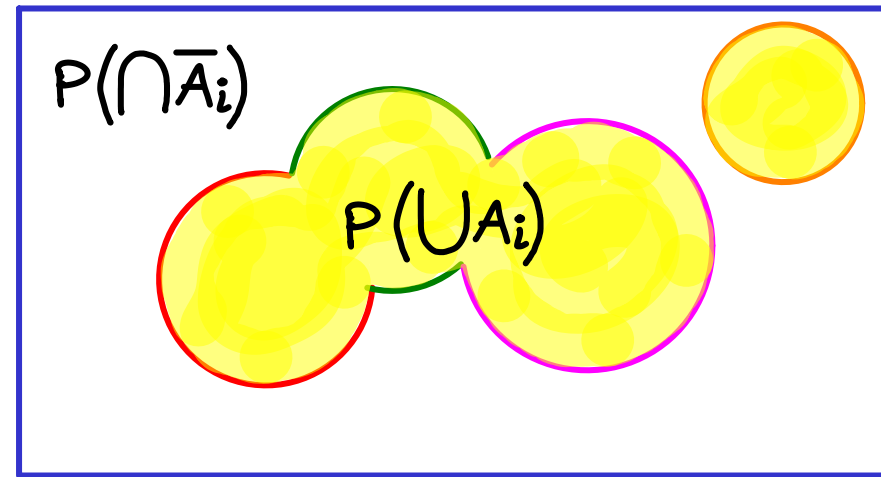
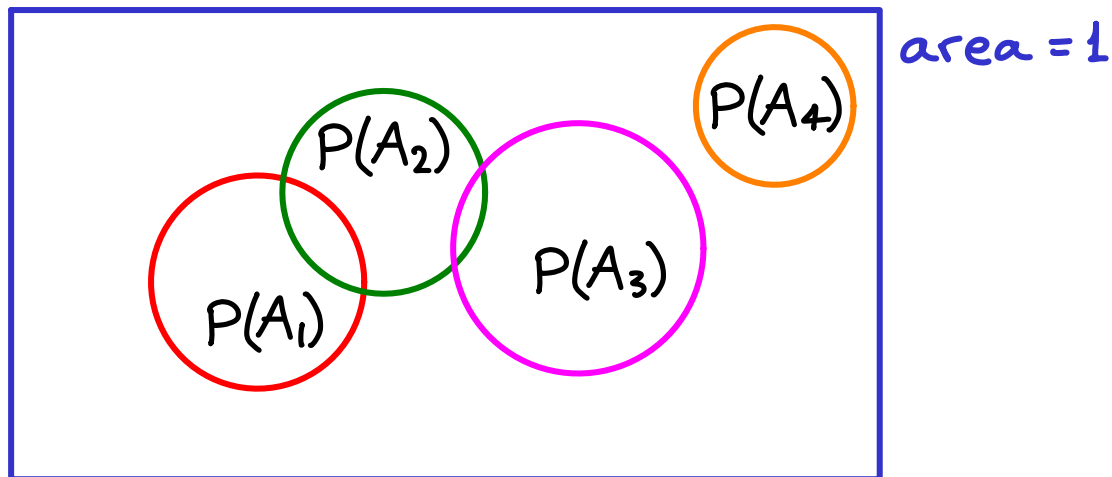
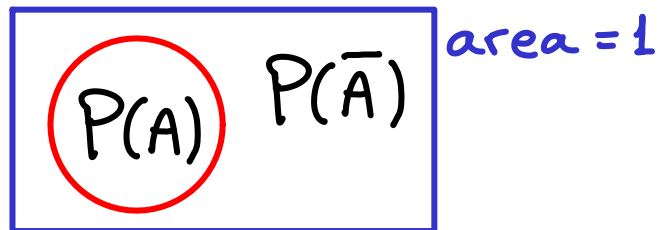


$$P(A) + P(\bar{A}) = 1$$



$$P(\cup A_i) = ?$$

$$P(A) + P(\bar{A}) = 1$$



$$P(\underbrace{\cup A_i}_{\text{inside any circle}}) = 1 - P(\underbrace{\cap \bar{A}_i}_{\text{outside every circle}})$$

$P(\text{someone in a group of size } k \text{ was born on April 1}) = ?$

$P(\text{someone in a group of size } k \text{ was born on April 1})$

$= P(\text{person 1 born on Apr. 1 OR person 2 born on Apr. 1 ...}$
 $\text{... OR person } k-1 \text{ born on Apr. 1 OR person } k \text{ born on Apr. 1})$

$P(\text{someone in a group of size } k \text{ was born on April 1})$

$= P(\text{person 1 born on Apr. 1 OR person 2 born on Apr. 1 ...}$
 $\text{... OR person } k-1 \text{ born on Apr. 1 OR person } k \text{ born on Apr. 1})$

Let e_i : person i born on Apr. 1

$P(\text{someone in a group of size } k \text{ was born on April 1})$

$= P(\text{person 1 born on Apr. 1 OR person 2 born on Apr. 1 ...}$
 $\text{... OR person } k-1 \text{ born on Apr. 1 OR person } k \text{ born on Apr. 1})$

Let e_i : person i born on Apr. 1 $\hookrightarrow P(e_1 \text{ OR } e_2 \text{ OR } e_3 \text{ ... OR } e_k)$

$P(\text{someone in a group of size } k \text{ was born on April 1})$

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Let e_i : person i born on Apr. 1 $\rightarrow P(\underline{e_1 \text{ OR } e_2 \text{ OR } e_3 \dots \text{ OR } e_k})$

$= P(\bigcup_{i=1}^k e_i)$

$P(\text{someone in a group of size } k \text{ was born on April 1})$

$= P(\text{person 1 born on Apr. 1 OR person 2 born on Apr. 1 ...}$
 $\text{... OR person } k-1 \text{ born on Apr. 1 OR person } k \text{ born on Apr. 1})$

Let e_i : person i born on Apr. 1 $\hookrightarrow P(e_1 \text{ OR } e_2 \text{ OR } e_3 \text{ ... OR } e_k)$

$$= P\left(\bigcup_{i=1}^k e_i\right) = 1 - P\left(\bigcap_{i=1}^k \overline{e_i}\right)$$

$P(\text{someone in a group of size } k \text{ was born on April 1})$

$= P(\text{person 1 born on Apr. 1 OR person 2 born on Apr. 1 ...}$
 $\text{... OR person } k-1 \text{ born on Apr. 1 OR person } k \text{ born on Apr. 1})$

Let e_i : person i born on Apr. 1 $\hookrightarrow P(e_1 \text{ OR } e_2 \text{ OR } e_3 \text{ ... OR } e_k)$

$$= P\left(\bigcup_{i=1}^k e_i\right) = 1 - P\left(\bigcap_{i=1}^k \overline{e_i}\right)$$

$= 1 - P(\text{person 1 NOT born on Apr. 1 AND person 2 NOT born on Apr. 1 ...}$

$\text{... person } k-1 \text{ NOT born on Apr. 1 AND person } k \text{ NOT born on Apr. 1})$

$P(\text{someone in a group of size } k \text{ was born on April 1})$

$= P(\text{person 1 born on Apr. 1 OR person 2 born on Apr. 1 ...}$
 $\text{... OR person } k-1 \text{ born on Apr. 1 OR person } k \text{ born on Apr. 1})$

Let e_i : person i born on Apr. 1 $\rightarrow P(e_1 \text{ OR } e_2 \text{ OR } e_3 \text{ ... OR } e_k)$

$$= P\left(\bigcup_{i=1}^k e_i\right) = 1 - P\left(\bigcap_{i=1}^k \overline{e_i}\right)$$

$= 1 - P(\text{person 1 NOT born on Apr. 1 AND person 2 NOT born on Apr. 1 ...}$

$\text{... person } k-1 \text{ NOT born on Apr. 1 AND person } k \text{ NOT born on Apr. 1})$

$= 1 - P(\text{nobody in a group of size } k \text{ was born on April 1})$

$$\begin{aligned} P(\text{someone in a group of } k \text{ was born on April 1}) &= P\left(\bigcup_{i=1}^k e_i\right) \\ &= 1 - P(\text{nobody in a group of } k \text{ was born on April 1}) = 1 - P\left(\bigcap_{i=1}^k \overline{e_i}\right) \end{aligned}$$

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minor approximation: assume no Feb. 29

$$P(\text{person } i \text{ not born on Apr. 1}) = \frac{364}{365} = P(\bar{e}_i)$$

(in practice, not all days are equally likely)

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(in practice, not all days are equally likely)

$$1 - P\left(\bigcap_{i=1}^k \overline{e_i}\right) = ?$$

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 P(\text{someone in a group of } k \text{ was born on April 1}) &= P\left(\bigcup_{i=1}^k e_i\right) \\
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$k = 80$
 $\sim 19.7\%$

$k = 365$
 $\sim 63.3\%$

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$$\begin{aligned}
 k &= 80 \\
 &\sim 19.7\%
 \end{aligned}$$

$$\begin{aligned}
 k &= 365 \\
 &\sim 63.3\%
 \end{aligned}$$

$$\left. \begin{array}{l} \text{approximation} \\ \text{(upper bound)} \end{array} \right\} P\left(\bigcup_{i=1}^k e_i\right) \leq \sum_{i=1}^k P(e_i)$$

$$P(\text{someone in a group of } k \text{ was born on April 1}) = P\left(\bigcup_{i=1}^k e_i\right)$$

$$= 1 - P(\text{nobody in a group of } k \text{ was born on April 1}) = 1 - P\left(\bigcap_{i=1}^k \bar{e}_i\right)$$

minor approximation: assume no Feb. 29

$$P(\text{person } i \text{ not born on Apr. 1}) = \frac{364}{365} = P(\bar{e}_i)$$

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$$k = 80 \\ \sim 19.7\%$$

$$k = 365 \\ \sim 63.3\%$$

approximation
(upper bound) } $P\left(\bigcup_{i=1}^k e_i\right) \leq \sum_{i=1}^k P(e_i) = k \frac{1}{365}$ $\sim 21.9\%$

$$P(\text{someone in a group of } k \text{ was born on April 1}) = P\left(\bigcup_{i=1}^k e_i\right)$$

$$= 1 - P(\text{nobody in a group of } k \text{ was born on April 1}) = 1 - P\left(\bigcap_{i=1}^k \bar{e}_i\right)$$

minor approximation: assume no Feb. 29

$$P(\text{person } i \text{ not born on Apr. 1}) = \frac{364}{365} = P(\bar{e}_i)$$

(in practice, not all days are equally likely)

$$1 - P\left(\bigcap_{i=1}^k \bar{e}_i\right) = 1 - \left(\frac{364}{365}\right)^k \rightarrow \text{because of independence}$$

$$k = 80 \\ \sim 19.7\%$$

$$k = 365 \\ \sim 63.3\%$$

approximation
(upper bound) } $P\left(\bigcup_{i=1}^k e_i\right) \leq \sum_{i=1}^k P(e_i) = k \frac{1}{365}$

$$\sim 21.9\%$$

bad
100%