

NON-CONSTRUCTIVE EXISTENCE PROOFS

(a short note)

Q: are there any odd numbers between 4 and 6 ?

A: yes, 5: $4 < 5 < 6$, and $5 = 2 \cdot 2 + 1$ (is odd)

Here the answer contains an example that can be verified

Sometimes we can prove that something exists without constructing an example or verification method.

Suppose that you can travel on a horizontal line,
varying your position & speed continuously.

You start at the origin at time $t=0$, move to the right,
and eventually return to the origin at time $t=1$.

Claim: at some time t' , $0 < t' < 1$, your speed will be zero.

Proof: To get back to the origin you must turn back,
in other words your speed goes from >0 to <0 . $\square$

We don't know where this happens, but it must happen.

Similar statements as the one we just saw:

- If the temperature at 7am was zero, and at noon was 10 degrees then at some time in the morning it was π degrees.

We don't know when, and we don't even know the exact value of π .

- At any given instant, there exist two positions on the equator that are diametrically opposite

(the center of the planet is exactly in between)

and that have exactly the same temperature.

(Assume perfectly spherical planet.)



Consider points A, B , diametrically opposite.

If temperatures are equal, done.

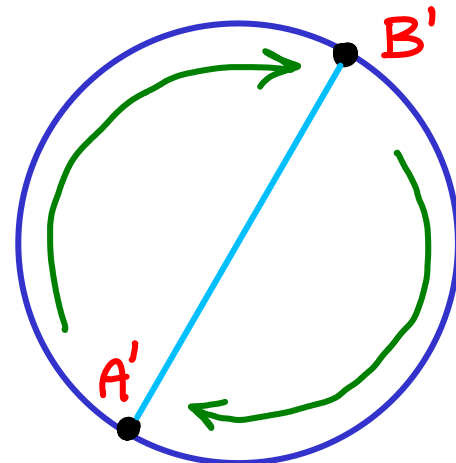
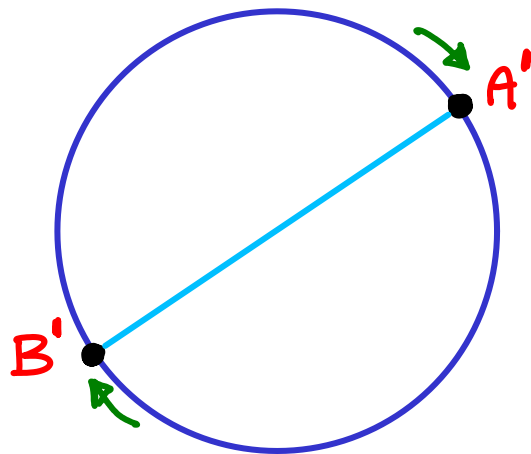
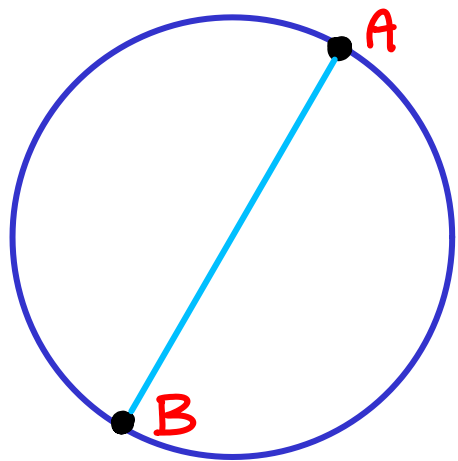
Else, without loss of generality, $t(A) > t(B)$.

Then move clockwise, staying opposite. Track points A', B' .

Continue while $t(A') > t(B')$. But when $B' = A$, $t(B') > t(A')$

So somewhere before rotating 180° , we had $t(A') = t(B')$. $\square$

All of this applies to one instant in time.



Note that there is no way to tell where the two points are.

More amazing fact: (Borsuk-Ulam theorem)

At any time there are 2 diametric points somewhere
on the planet (not necessarily on the equator)
that have equal temperature AND equal pressure.

(works for any 2 continuous functions)

There exist two irrational numbers a, b ,
such that a^b is rational.

Recall that $\sqrt{2}$ is irrational. Consider $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$.

If $(\sqrt{2}^{\sqrt{2}})$ is rational then $a = b = \sqrt{2}$.

Otherwise, use $(x^y)^z = x^{yz}$: $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$

$$a = \sqrt{2}^{\sqrt{2}}, \quad b = \sqrt{2}. \quad \square$$

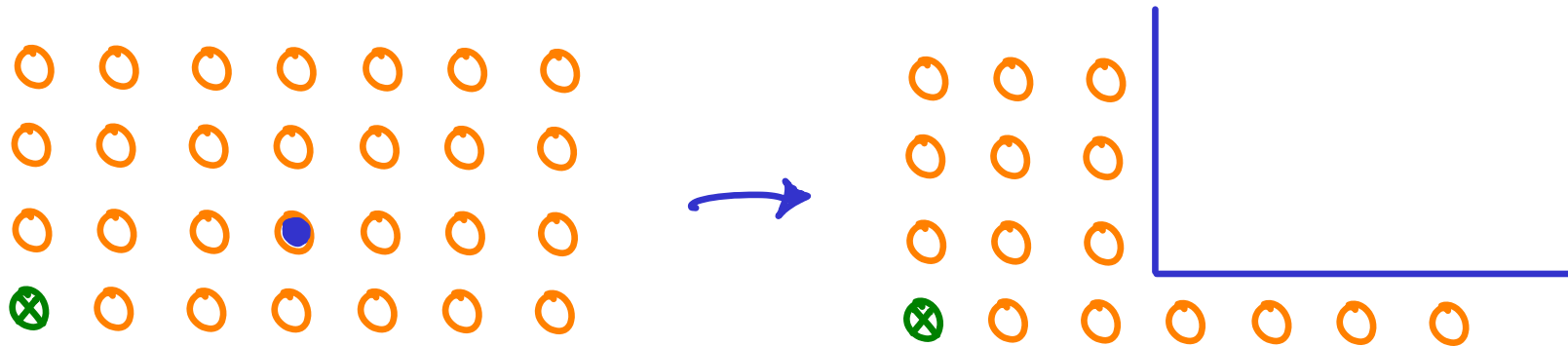
This proof doesn't tell us exactly what a is, but it exists.

The game of Chomp

2 players take turns eating cookies from a grid.

↳ If you take some cookie then you must also take all cookies above and/or to the right.

Whoever eats the poisoned cookie at the bottom-left loses.



Claim:
there exists a
winning strategy
for player 1.

Suppose player 1 eats only the top-right cookie.

There are 2 possibilities after doing so:

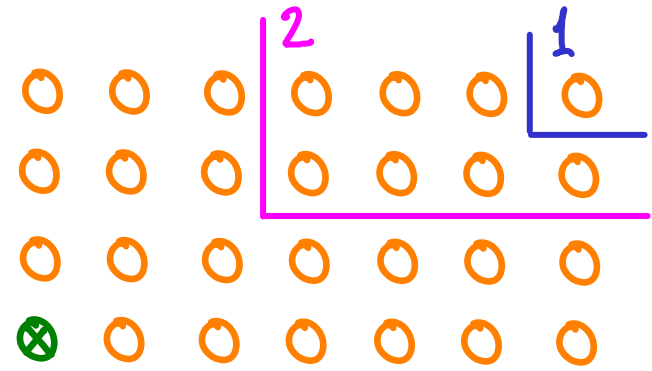
A) there exists a winning strategy for player 1
no matter what player 2 does. (claim proved)

B) there exists a winning strategy for player 2.

↳ Then player 2 has some first move that ensures a win.

But then player 1 could have made that move and won. $\square$

This proof offers no strategy. No general solution is known.



Other non-constructive proofs:

- A few pigeonhole proofs

e.g., see proof about disjoint subsets with same sum.

- Every integer $n \geq 2$ is a product of ≥ 2 primes.

See notes on Induction