Concepts to be familiar with before reading this document

$\neg X$: "not X"

"proposition"

IF-THEN $\rightarrow$

IFF $\leftrightarrow$

TRUTH TABLES

If P is a proposition then so is $\neg P$.

Any proposition is either true (T) or false (F)

but P and $\neg P$ can't both be T, or both F.

If we know one, we know the other

P	$\neg P$
T	F
F	T

Recall: "it's raining" and "it's cloudy" are not propositions. ← can vary

"if it's raining then it's cloudy" is a proposition. It's always true.

"if it's raining then it's not cloudy" is a proposition. It's always false.

P = "it's raining"   Q = "it's cloudy"

P & Q are Boolean variables aka propositional variables

George Boole, 1840's

that can be used in other statements, e.g., (P AND Q)

"it is raining and cloudy"

P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

only one way for this to happen

P = "it's raining"
Q = "it's cloudy"

P OR Q

"it is $\underbrace{\text{raining or cloudy}}_{\text{don't care which}}$"

P	Q	P OR Q
T	T	T
T	F	T
F	T	T
F	F	F

this combination can't happen!
We know that $P \rightarrow Q$ in our example.

Is the truth table wrong or invalid?

Could have asked the same for P AND Q

No. This is the truth table for OR.

All combinations are considered. Context & extra info isn't.

$\underbrace{P \text{ OR } Q}$

don't care which is true

P	Q	P OR Q
T	T	T
T	F	T
F	T	T
F	F	F

compare this to:

- "either P or Q but not both"
- "precisely one of P, Q"
- "exclusively one of P or Q"
- P XOR Q

P	Q	P XOR Q
T	T	F
T	F	T
F	T	T
F	F	F

P IFF Q

$P \leftrightarrow Q$

$P \rightarrow Q$ and $Q \rightarrow P$

P	Q	P IFF Q	
T	T	T	consistent
T	F	F	inconsistent
F	T	F	inconsistent
F	F	T	

Example:

P: pigs can fly (F)

Q: dogs can talk (F)

P IFF Q: This is true! (when P, Q are F)

if pigs can fly then dogs can talk
and
if dogs can talk then pigs can fly

$P \rightarrow Q$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

consistent
inconsistent

Example:

P: pigs can fly (F)

Q: I like apples F? T?

$P \rightarrow Q$: This is true! (when P is F)

if pigs can fly then I like apples

Vacuous truth

It doesn't matter if I like apples or not

If it doesn't matter / apply, why bother considering these cases?

It is often important to simplify statements with Boolean variables

Example coming soon

$P \to Q$

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

consistent
inconsistent

Example:

P: pigs can fly (F)

Q: I like apples F? T?

$P \to Q$: This is true! (when P is F)

if pigs can fly then I like apples

It doesn't matter if I like apples

Example 2:

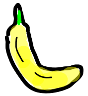
P: $\exists$ life on Mars F? T?

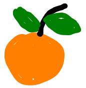
Q: I like basketball (T)

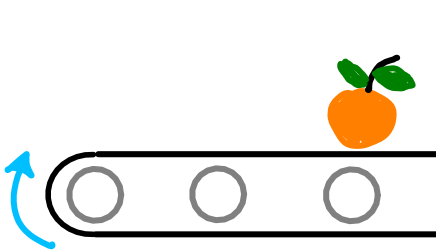
$P \to Q$: This is true! (when Q is T)

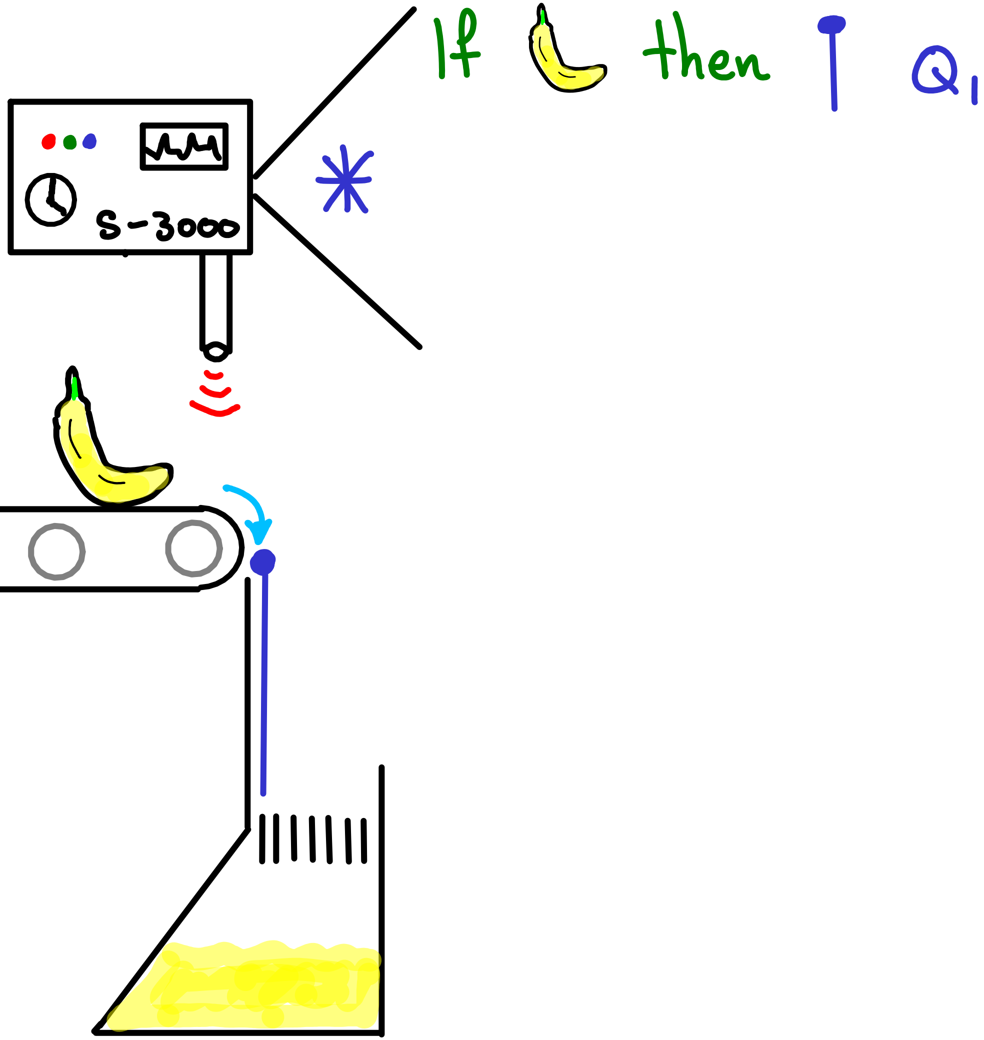
if $\exists$ life on Mars then I like basketball

It doesn't matter if there is life on Mars

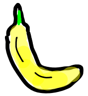
A machine has a sensor that sets 2 variables. If it senses  then $P_1 = T$, otherwise $P_1 = F$.

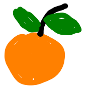
If it senses  then $P_2 = T$, otherwise $P_2 = F$.



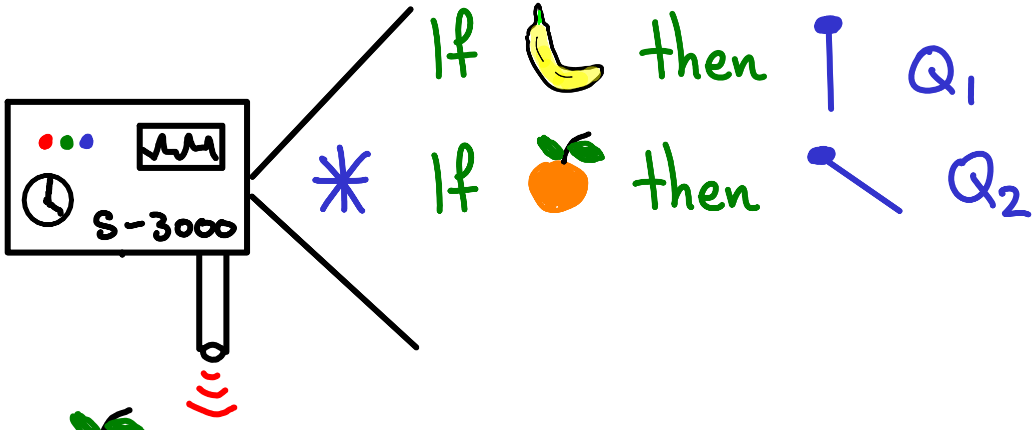


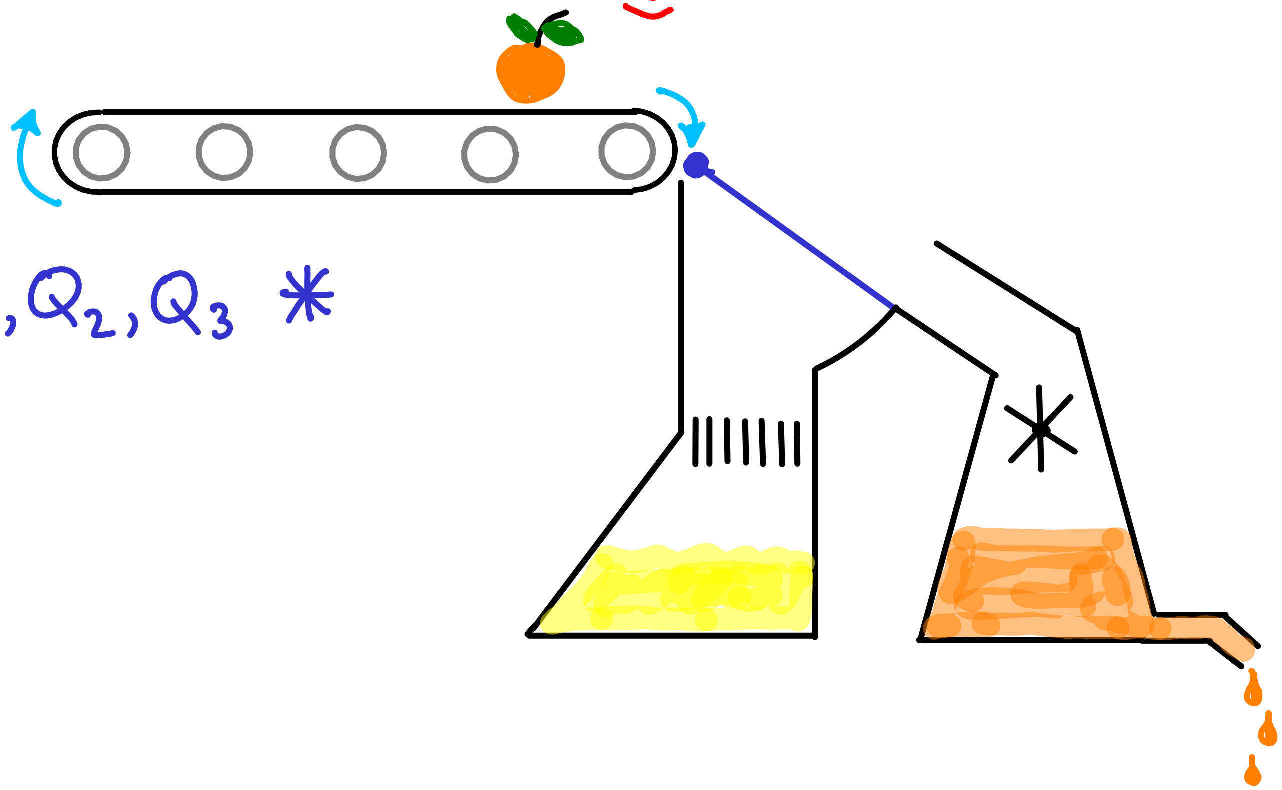
Also, 3 desired actions, Q_1, Q_2, Q_3 *

A machine has a sensor that sets 2 variables. If it senses  then $P_1 = T$, otherwise $P_1 = F$.

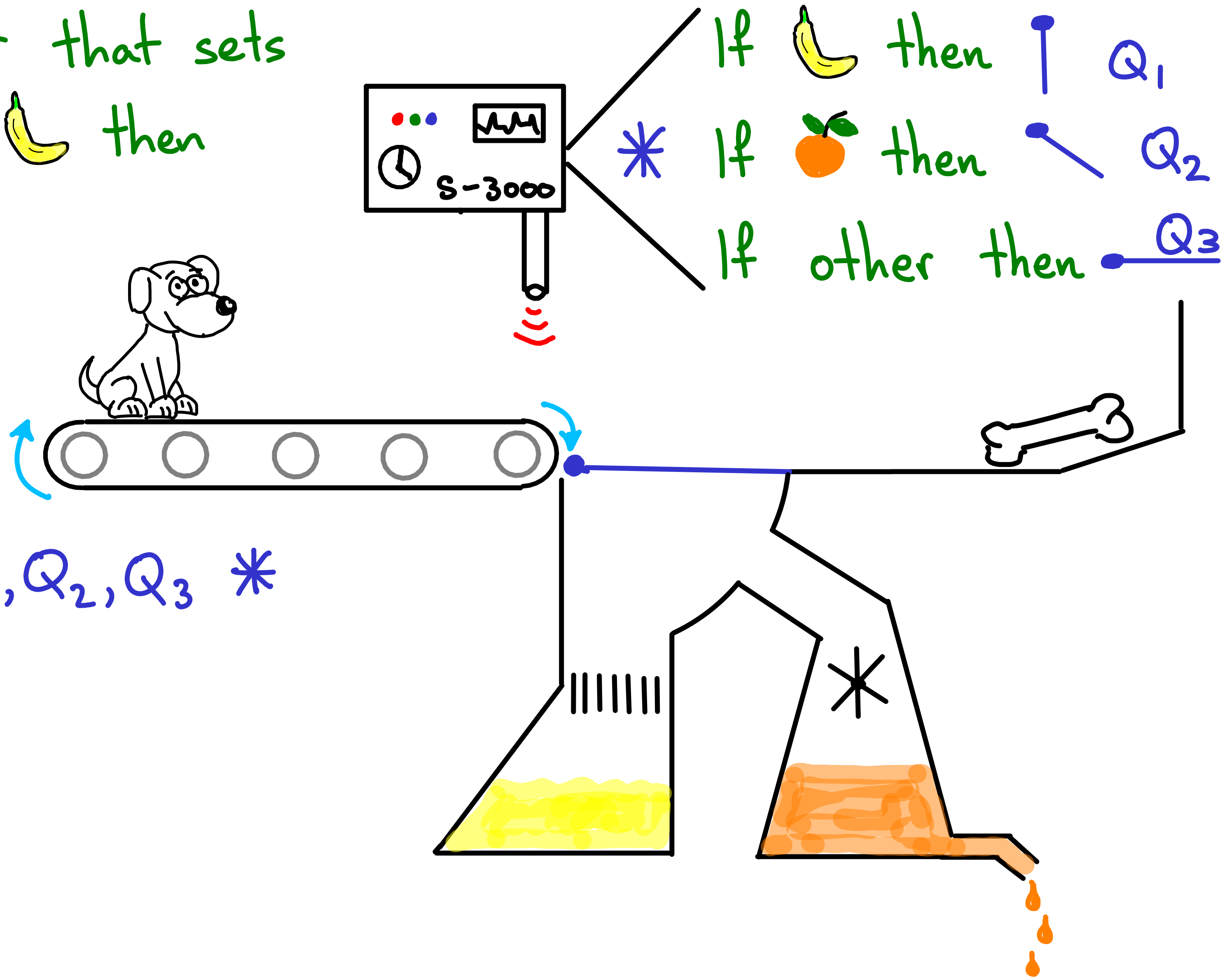
If it senses  then $P_2 = T$, otherwise $P_2 = F$.

Also, 3 desired actions, Q_1, Q_2, Q_3 $*$





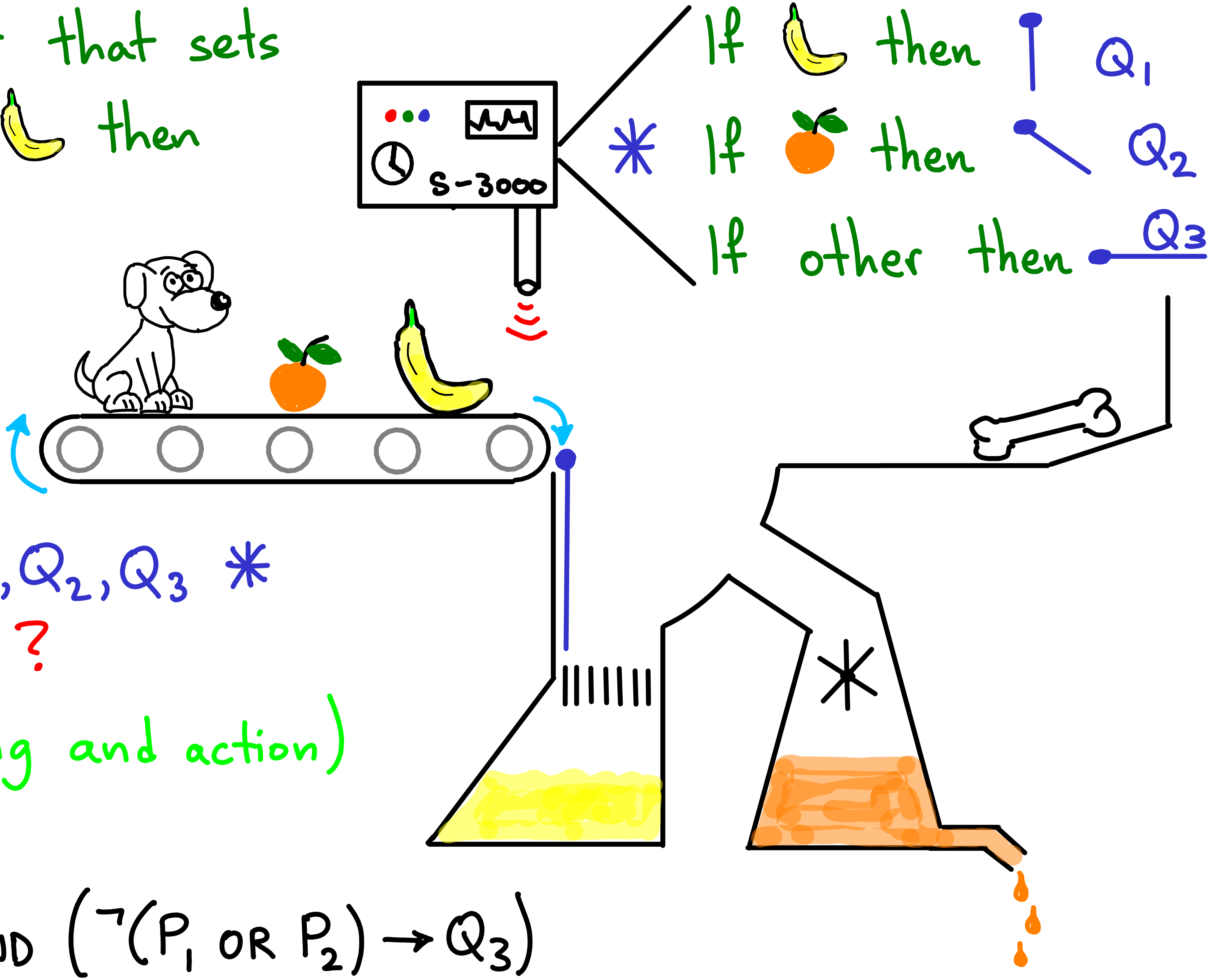
A machine has a sensor that sets 2 variables. If it senses then $P_1 = T$, otherwise $P_1 = F$. If it senses then $P_2 = T$, otherwise $P_2 = F$.



Also, 3 desired actions, Q_1, Q_2, Q_3 *

A machine has a sensor that sets 2 variables. If it senses 🍌 then $P_1 = T$, otherwise $P_1 = F$.

If it senses 🍊 then $P_2 = T$, otherwise $P_2 = F$.



Also, 3 desired actions, Q_1, Q_2, Q_3 *

Is the machine working?

(given a sensor reading and action)

Evaluate:

$$(P_1 \rightarrow Q_1) \text{ AND } (P_2 \rightarrow Q_2) \text{ AND } (\neg(P_1 \text{ OR } P_2) \rightarrow Q_3)$$

P_1 IFF . P_2 IFF .

Is the machine working? Yes.

e.g., sense , action =

$(P_1 \to Q_1)$ } F → F : T

AND

$(P_2 \to Q_2)$ } T → T : T

AND

$(\neg(P_1 \text{ OR } P_2) \to Q_3)$ } ¬(F OR T) → F

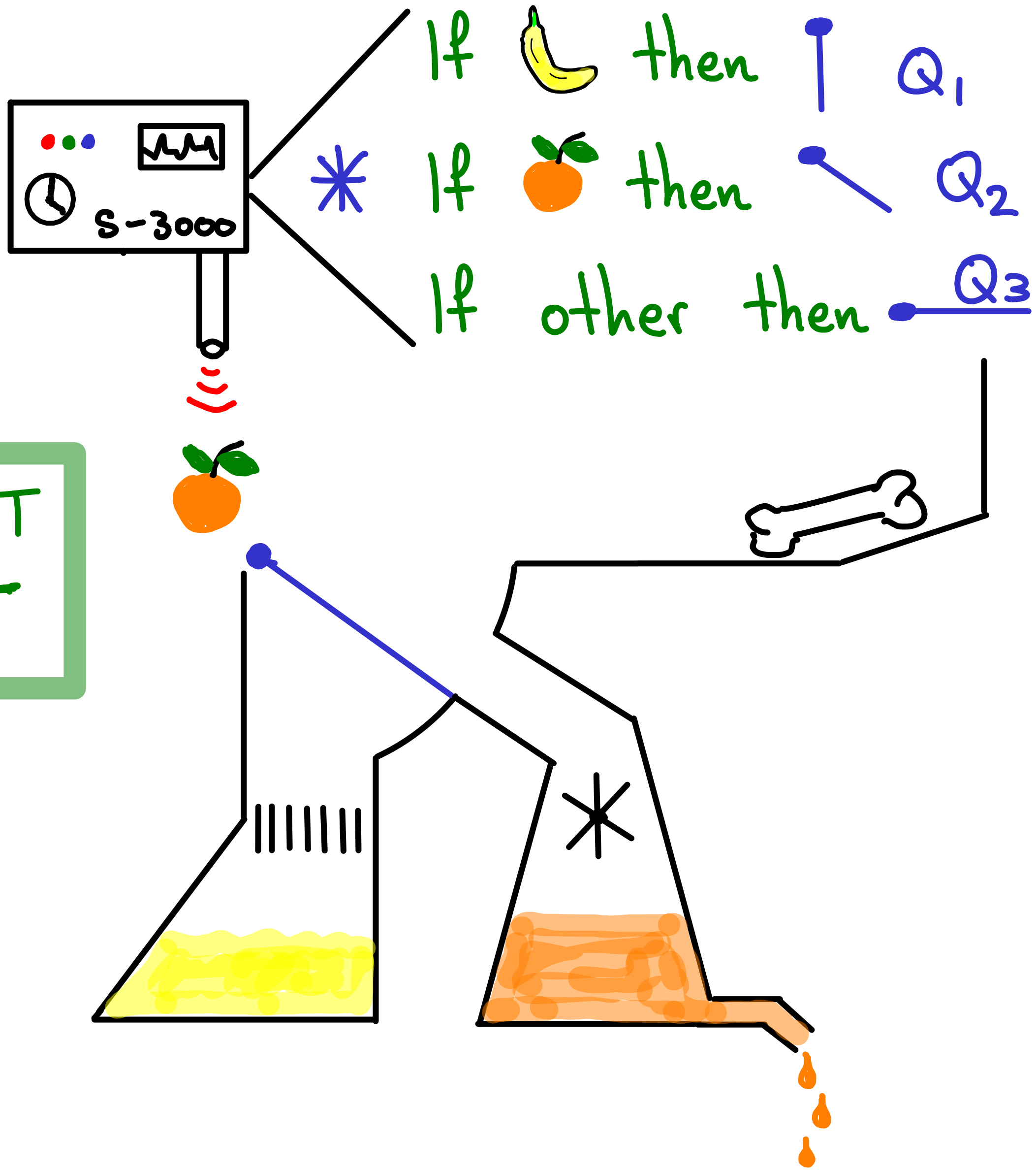
¬T → F

F → F

T

T AND T AND T

conclusion: T



P	Q	P → Q	P OR Q
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

P_1 IFF . P_2 IFF .

Is the machine working? No

e.g., sense , action =

$(P_1 \rightarrow Q_1)$ } $T \rightarrow F$: F

AND

$(P_2 \rightarrow Q_2)$ } $F \rightarrow F$: T

AND

$(\neg(P_1 \text{ OR } P_2) \rightarrow Q_3)$ } $\neg(T \text{ OR } F) \rightarrow T$

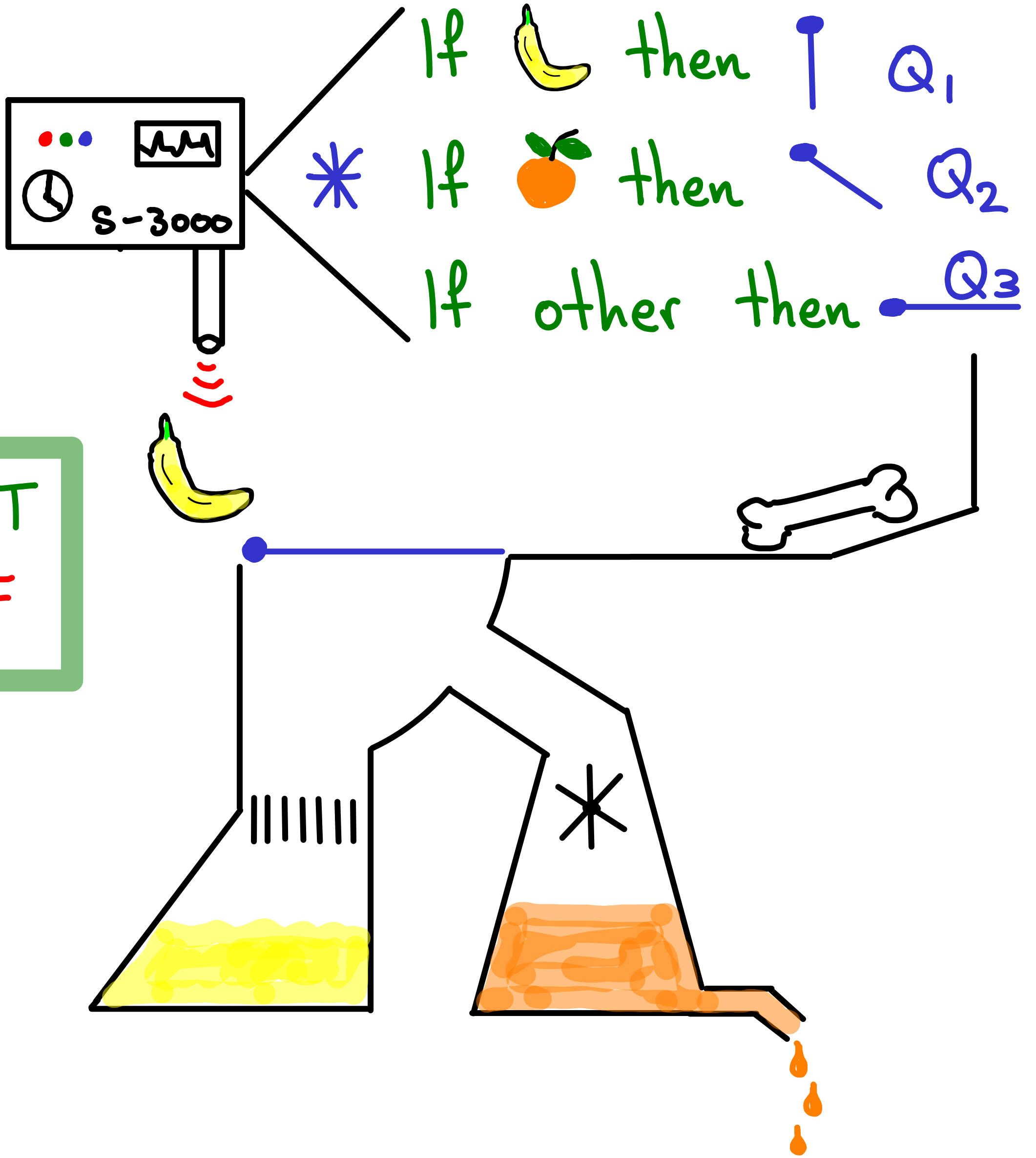
$\neg T \rightarrow T$

$F \rightarrow T$

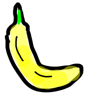
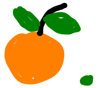
T

F AND T AND T

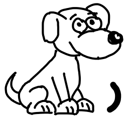
conclusion: F



P	Q	P → Q	P OR Q
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

P_1 IFF . P_2 IFF .

Is the machine working? No

e.g., sense , action =

$(P_1 \to Q_1)$ } F → T : T

AND

$(P_2 \to Q_2)$ } F → F : T

AND

$(\neg(P_1 \text{ OR } P_2) \to Q_3)$ }

¬(F OR F) → F

¬F → F

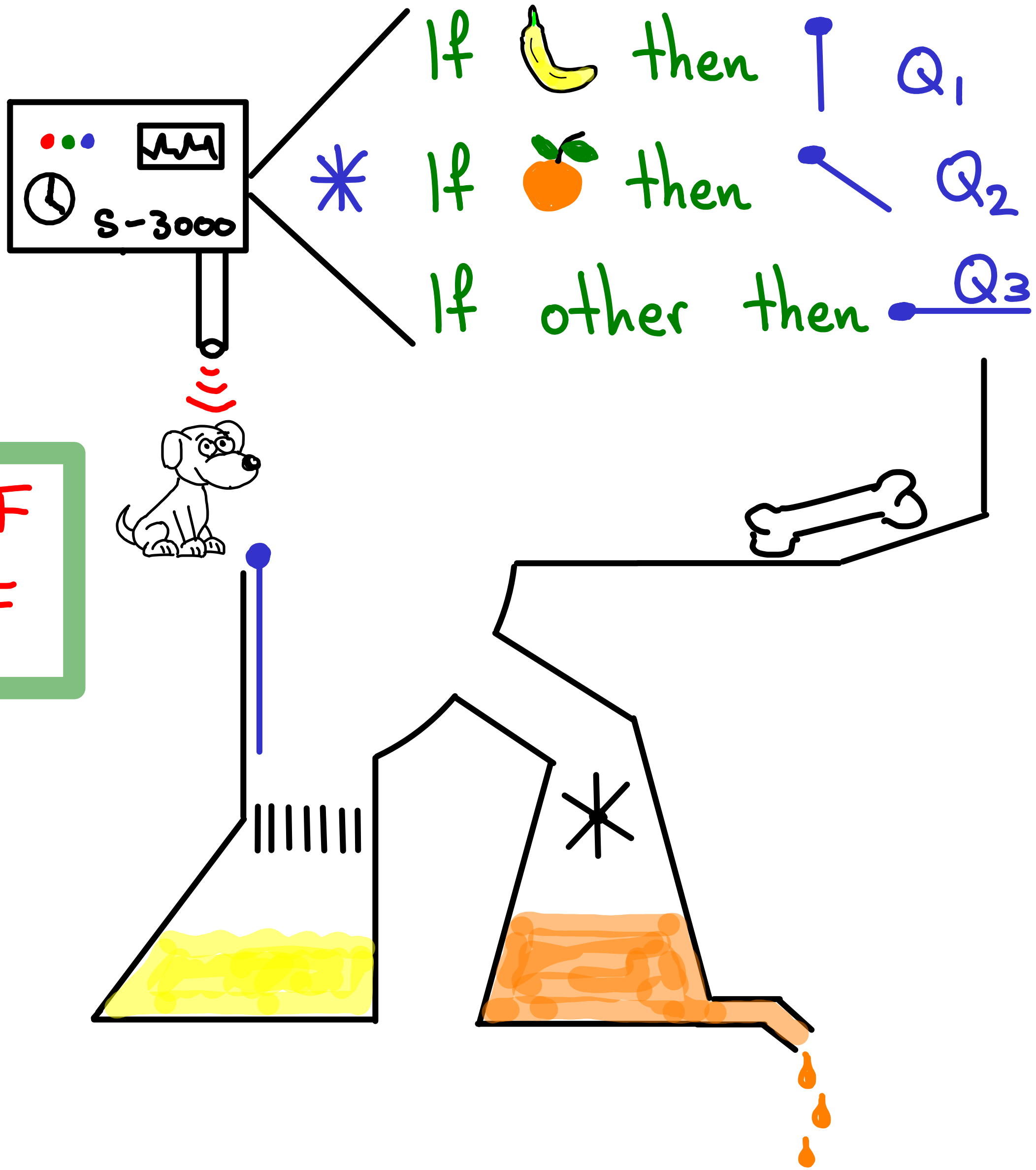
T → F

F

T AND T AND F

conclusion: F

P	Q	P → Q	P OR Q
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F



PROPOSITIONAL LOGIC NOTATION

	NOT P	$\neg P$ or $\overline{P}$
	P AND Q	$P \wedge Q$
	P OR Q	$P \vee Q$
if P then Q,	P implies Q	$P \rightarrow Q$
	P IFF Q	$P \leftrightarrow Q$
	P XOR Q	$P \oplus Q$

MCS: "cryptic ... we mostly stick to words"

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$\neg Q \rightarrow \neg P$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

If I am hungry then I eat

$P \rightarrow Q$

contrapositive

If I don't eat then I am not hungry

$\neg Q \rightarrow \neg P$

If I eat then I am hungry

$Q \rightarrow P$

converse

If a logic formula is always T then it is valid.

$$\underbrace{B \text{ OR } \neg B}_{T}$$

See also: tautology

B	$\neg B$	B OR $\neg B$
T	F	T
F	T	T

B XOR $\neg B$ is valid:

B	$\neg B$	B XOR $\neg B$
T	F	T
F	T	T

If you're asked: "do you want cake now or later?" ... just say YES

Also works with: "do you want cake or ice cream?"

If a logic formula can be T then it is satisfiable

P is satisfiable IFF $\neg$P is not valid

We have seen that a valid formula (tautology) can be simplified

Let's look at some more ways to simplify logic (formulas)

F AND $A \longleftrightarrow F$

T AND $A \longleftrightarrow A$

T OR $A \longleftrightarrow T$

F OR $A \longleftrightarrow A$

A AND $\neg A \longleftrightarrow F$

A OR $\neg A \longleftrightarrow T$

A AND $A \longleftrightarrow A$

A OR $A \longleftrightarrow A$

$$A \longleftrightarrow \neg(\neg A)$$

P: A OR $(\neg A$ AND $B)$ $\longleftrightarrow$ A OR B

2 options

- if A then P regardless of what $(\neg A$ AND $B)$ is.

OR

- if $\neg A$, then we need $(\neg A$ AND $B)$, to get P.

But this is true in this case

So we need $(T$ AND $B)$, i.e., we need B

$$\underbrace{A}\ \text{OR}\ (\neg A\ \text{AND}\ \underbrace{B}) \longleftrightarrow A\ \text{OR}\ B$$

if $x<0$ do [action C]
else if $(x \geq 0$ and $y>10)$ do [action C]

more efficient →

if $\underbrace{x<0}$ do [action C]
else if $\underbrace{y>10}$ do [action C]

$$A \text{ OR } (\neg A \text{ AND } B) \quad \longleftrightarrow \quad A \text{ OR } B$$

A	B	A OR (¬A AND B)	A OR B
T	T	T	T
T	F	T	T
F	T	T	T
F	F	$\underbrace{\text{F OR (T AND F)}}_{\text{F}}$	F

SIMPLIFYING PROPOSITIONAL FORMULAS

Can replace $A \to B$ with $\neg A$ OR B

vacuous $A \to B$

because A

Recall, $\underbrace{\neg A}_{\text{case 1}}$ OR $(\underbrace{A}_{\text{case 2}}$ AND $B)$

SIMPLIFYING PROPOSITIONAL FORMULAS

Can replace $A \rightarrow B$ with $\neg A$ OR B

Can replace $A \leftrightarrow B$ with $(A \rightarrow B)$ AND $(B \rightarrow A)$, then

$(\neg A$ OR $B)$ AND $(\neg B$ OR $A)$

Can replace A XOR B with $(A$ OR $B)$ AND $\neg(A$ AND $B)$

So we can get everything in terms of AND, OR, NOT.

Next we see several rules that can help to simplify/modify further

$A \text{ AND } B \leftrightarrow B \text{ AND } A$

$A \text{ OR } B \leftrightarrow B \text{ OR } A$

commutativity

$(A \text{ AND } B) \text{ AND } C \leftrightarrow A \text{ AND } (B \text{ AND } C)$

$A \text{ AND } B \text{ AND } C$

$(A \text{ OR } B) \text{ OR } C \leftrightarrow A \text{ OR } (B \text{ OR } C)$

$A \text{ OR } B \text{ OR } C$

associativity

$A \text{ AND } (B \text{ OR } C) \leftrightarrow (A \text{ AND } B) \text{ OR } (A \text{ AND } C)$

$A \text{ OR } (B \text{ AND } C) \leftrightarrow (A \text{ OR } B) \text{ AND } (A \text{ OR } C)$

distributivity

$$\neg(A \text{ AND } B) \longleftrightarrow \neg A \text{ OR } \neg B$$

$$\neg(A \text{ OR } B) \longleftrightarrow \neg A \text{ AND } \neg B$$

De Morgan's Laws

not (rich and famous) <---> (not rich) or (not famous)

not (fast or strong) <---> (not fast) and (not strong)

$(A \to B)$ AND $(B \to A)$ $\stackrel{?}{=}$ $(A$ AND $B)$ OR $(\neg A$ AND $\neg B)$

$=$ $(\neg A$ OR $B)$ AND $(\neg B$ OR $A)$

by earlier result

Can replace $A \to B$ with $\neg A$ OR B

vacuous $A \to B$

because A

Recall, $\underbrace{\neg A}_{\text{case 1}}$ OR $(\underbrace{A}_{\text{case 2}}$ AND $B)$

$$(A \to B) \text{ AND } (B \to A) \stackrel{?}{=} (A \text{ AND } B) \text{ OR } (\neg A \text{ AND } \neg B)$$

$$= (\neg A \text{ OR } B) \text{ AND } (\neg B \text{ OR } A) \qquad \text{by earlier result}$$

$$= (\neg A \text{ AND } (\neg B \text{ OR } A)) \text{ OR } (B \text{ AND } (\neg B \text{ OR } A)) \qquad \text{distr.}$$

treat $(\neg B \text{ OR } A)$ as C

$$(A \to B) \text{ AND } (B \to A) \overset{?}{=} (A \text{ AND } B) \text{ OR } (\neg A \text{ AND } \neg B)$$

$$= (\neg A \text{ OR } B) \text{ AND } (\neg B \text{ OR } A)$$ by earlier result

$$= (\neg A \text{ AND } (\neg B \text{ OR } A)) \text{ OR } (B \text{ AND } (\neg B \text{ OR } A))$$ distr.

$$= ((\neg A \text{ AND } \neg B) \text{ OR } (\neg A \text{ AND } A)) \text{ OR } ((B \text{ AND } \neg B) \text{ OR } (B \text{ AND } A))$$ »

$$= ((\neg A \text{ AND } \neg B) \text{ OR } F) \text{ OR } (F \text{ OR } (B \text{ AND } A))$$

$$= (\neg A \text{ AND } \neg B) \text{ OR } F \text{ OR } F \text{ OR } (B \text{ AND } A)$$ assoc.

$$= (\neg A \text{ AND } \neg B) \text{ OR } (B \text{ AND } A)$$

$$= (A \text{ AND } B) \text{ OR } (\neg A \text{ AND } \neg B)$$ comm.

DISJUNCTIVE FORM "an OR of ANDs"

e.g., (A AND B) OR (A AND C AND D) OR ($\bar{B}$ AND D)

CONJUNCTIVE FORM "an AND of ORs"

e.g., (A OR C) AND (A OR $\bar{D}$) AND ($\bar{B}$ OR C OR D)

DISJUNCTIVE FORM "an OR of ANDs"

e.g., (A AND B) OR (A AND C AND D) OR ($\overline{B}$ AND D)

We can write any propositional formula like this. e.g., A AND (B OR C)

A	B	C	A AND (B OR C)
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

Fill in entire truth table :(

Find all rows in table where formula is T.

Want ≥1 of these rows to be satisfied:

(A AND B AND C)

OR (A AND B AND $\overline{C}$)

OR (A AND $\overline{B}$ AND C)

CONJUNCTIVE FORM "an AND of ORs"

e.g., $(A \text{ OR } C) \text{ AND } (A \text{ OR } \bar{D}) \text{ AND } (\bar{B} \text{ OR } C \text{ OR } D)$

We can write any propositional formula like this. e.g., $A \text{ AND } (B \text{ OR } C)$

A	B	C	A AND (B OR C)
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

Fill in entire truth table

Find all rows in table where formula is F.

Want ALL of these rows to NOT be satisfied:

$$(\bar{A} \text{ OR } B \text{ OR } C)$$
$$\text{AND } (A \text{ OR } \bar{B} \text{ OR } \bar{C})$$
$$\text{AND } (A \text{ OR } \bar{B} \text{ OR } C)$$
$$\text{AND } (A \text{ OR } B \text{ OR } \bar{C})$$
$$\text{AND } (A \text{ OR } B \text{ OR } C)$$

What we got was actually DISJUNCTIVE NORMAL FORM

& CONJUNCTIVE NORMAL FORM

Every variable is present in each term within parentheses.

Why do we care about this?

This standardization can help proof automation (for validity, satisfiability)

You could also check if two formulas are equivalent,
by getting them into the same "canonical" form

Source: MCS

Other sources use "CNF" and "DNF" without this requirement

What we got was actually DISJUNCTIVE NORMAL FORM

& CONJUNCTIVE NORMAL FORM

Every variable is present in each term within parentheses.

We can often simplify DNF, CNF

$(A \text{ AND } B \text{ AND } C)$
OR $(A \text{ AND } B \text{ AND } \overline{C})$
OR $(A \text{ AND } \overline{B} \text{ AND } C)$

$(A \text{ AND } B)$
OR $(A \text{ AND } C)$

e.g.,

A	B	C	A AND (B OR C)
T	T	T	T
T	T	F	T
T	F	T	T

"for all" $\forall$ vs $\exists$ "exists"

$\forall x \in \mathbb{R}.\ x^2 \geq 0$ $\exists x \in \mathbb{R}.\ x - \pi^2 + \sqrt{2} = 0$

For every action there is a reaction. $\forall a \exists r \neq \exists r \forall a$ *

There is an answer for every question.

Ambiguous → One answer for all questions? $\exists a \forall q$

$\neq$

For every question there is an answer. $\forall q \exists a$

---

* Inconsistency in literature

Every coin has two sides: $\forall c\, \exists s_1(c)\, \exists s_2(c)$

P = prime numbers. X = even integers > 2.

$\forall n \in X\, \exists a \in P\, \exists b \in P.\; n = a + b$

For every integer n greater than 2,
there exist prime numbers a and b such that $n = a + b$.

Every integer greater than 2 is the sum of two primes.
(Goldbach's conjecture)

$\forall n \in X \, \exists a \in P \, \exists b \in P. \; n = a + b$

Every integer greater than 2 is the sum of two primes.

$\forall a \in P \, \forall b \in P \, \exists n \in X. \; n = a + b$

For every pair of primes there is an integer >2 that is their sum.

$\exists a \in P \, \exists b \in P. \, \forall n \in X \; n = a + b$

There exist 2 primes such that their sum is equal to every integer >2

$\exists a \in P \, \exists b \in P \, \forall n \in X. \; n = a + b$

(poor form)

Every integer greater than 2 is the sum of two primes.

$\forall x. P(x)$ For all x, proposition P (with x as a $\underbrace{\text{variable}}_{\text{that makes sense for P}}$) is true.

e.g., P(Alex) : Alex likes logic.

Not everybody likes logic. $\neg(\forall x. P(x))$

There is someone who doesn't like logic. $\exists x. \neg(P(x))$

actually not true.

Nobody can lick their own elbow. $\neg(\exists x. P(x))$

Every person can't lick their own elbow. $\forall x. \neg(P(x))$