It is assumed that you know:

$\exists$, $\mathbb{Z}$, $\mathbb{N}$

$$\text{even} + \text{even} = \text{even}$$
$$\text{odd} + \text{odd} = \text{even}$$
$$\text{even} + \text{odd} = \text{odd}$$

iff

$$\sum_{x=1}^{0} x = 0$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$
$$0! = 1$$

not critical

n-choose-k: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

PROOF by INDUCTION

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Suppose you have n dominoes in a row.

FACT: If the k-th domino falls to the right,
then so does the (k+1)-st

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FACT: If the k-th domino falls, then so does the (k+1)-st

Apply FACT: if the 1st domino falls, then so does the 2nd.

Claim: if we tip the 1st domino, then eventually the n-th falls

FACT: If the k-th domino falls, then so does the (k+1)-st

Apply FACT: if the 1st domino falls, then so does the 2nd.

(again) ... if the 2nd domino falls, then so does the 3rd.

Claim: if we tip the 1st domino, then eventually the n-th falls

FACT: If the k-th domino falls, then so does the (k+1)-st

Apply FACT: if the 1st domino falls, then so does the 2nd.

(again) ... if the 2nd domino falls, then so does the 3rd.

...

...

... if the (n-1)-st domino falls, then so does the n-th.

Claim: if we tip the 1st domino, then eventually the n-th falls

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Assuming that our claim is true for n-1...

Claim: if we tip the 1st domino, then eventually the n-th falls

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"if we tip the 1st domino, then eventually the (n-1)-st falls"

Claim: if we tip the 1st domino, then eventually the n-th falls

FACT: If the k-th domino falls, then so does the (k+1)-st

Assuming that our claim is true for n-1...

"if we tip the 1st domino, then eventually the (n-1)-st falls"

Apply FACT once: if the (n-1)-st domino falls, then so does the n-th.

Claim: if we tip the 1st domino, then eventually the n-th falls

FACT: If the k-th domino falls, then so does the (k+1)-st

Assuming that our claim is true for n-1...

"if we tip the 1st domino, then eventually the (n-1)-st falls"

Apply FACT once: if the (n-1)-st domino falls, then so does the n-th.

...we prove original claim.

Claim: if we tip the 1st domino, then eventually the n-th falls

assuming our claim is true for n-1, then...

we prove claim for n

Claim: if we tip the 1st domino, then eventually the n-th falls

assuming our claim is true for n-2, then...

~~assuming~~ our claim is true for n-1, then...

we prove claim for n

Claim: if we tip the 1st domino, then eventually the n-th falls

assuming our claim is true for 2, then...

⋮

~~assuming~~ our claim is true for n-2, then...

~~assuming~~ our claim is true for n-1, then...

we prove claim for n

Claim: if we tip the 1st domino, then eventually the n-th falls

Using FACT,

~~assuming~~ our claim is true for 2, then...

⋮

~~assuming~~ our claim is true for n-2, then...

~~assuming~~ our claim is true for n-1, then...

we prove claim for n

PROOF by INDUCTION

You want to prove something involving n. "n will fall"

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PROOF by INDUCTION

You want to prove something involving n. "n will fall"

- Assume that it's true if you have n-1 instead. "n-1 will fall"
- Prove that this assumption helps solve the actual problem. $n-1 \rightarrow n$
- Don't forget to push the first domino!
  - ↳ Prove what you need for a small value $1 \rightarrow 2$

PROOF by INDUCTION

You want to prove something involving n: $\sum_{i=0}^{n} 3^i < 3^{n+1}$

PROOF by INDUCTION

You want to prove something involving n: $\sum_{i=0}^{n} 3^i < 3^{n+1}$

- Assume that it's true if you have n-1 : $\rightarrow$?

PROOF by INDUCTION

You want to prove something involving n: $\sum_{i=0}^{n} 3^i < 3^{n+1}$

- Assume that it's true if you have n-1 : $\sum_{i=0}^{n-1} 3^i < 3^{(n-1)+1}$

PROOF by INDUCTION

You want to prove something involving n: $\sum_{i=0}^{n} 3^i < 3^{n+1}$

- Assume that it's true if you have n-1 : $\sum_{i=0}^{n-1} 3^i < 3^{(n-1)+1}$

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...?

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- Prove that this assumption helps solve the actual problem.

$$\sum_{i=0}^{n} 3^i = 3^n + \sum_{i=0}^{n-1} 3^i \quad \ldots ?$$

PROOF by INDUCTION

You want to prove something involving n: $\sum_{i=0}^{n} 3^i < 3^{n+1}$

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- Prove that this assumption helps solve the actual problem.

$$\sum_{i=0}^{n} 3^i = 3^n + \sum_{i=0}^{n-1} 3^i < 3^n + 3^{(n-1)+1} \quad \ldots ?$$

PROOF by INDUCTION

You want to prove something involving n: $\sum_{i=0}^{n} 3^i < 3^{n+1}$

- Assume that it's true if you have n-1 : $\sum_{i=0}^{n-1} 3^i < 3^{(n-1)+1}$

- Prove that this assumption helps solve the actual problem.

$$\sum_{i=0}^{n} 3^i = 3^n + \sum_{i=0}^{n-1} 3^i < 3^n + 3^{(n-1)+1} = 3^n + 3^n < 3 \cdot 3^n = 3^{n+1}$$

PROOF by INDUCTION

You want to prove something involving n: $\sum_{i=0}^{n} 3^i < 3^{n+1}$

- Assume that it's true if you have n-1 : $\sum_{i=0}^{n-1} 3^i < 3^{(n-1)+1}$
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- Prove what you need for a small value ... ?

PROOF by INDUCTION

You want to prove something involving n: $\sum_{i=0}^{n} 3^i < 3^{n+1}$

- Assume that it's true if you have n-1 : $\sum_{i=0}^{n-1} 3^i < 3^{(n-1)+1}$

- Prove that this assumption helps solve the actual problem.

$$\sum_{i=0}^{n} 3^i = 3^n + \sum_{i=0}^{n-1} 3^i < 3^n + 3^{(n-1)+1} = 3^n + 3^n < 3 \cdot 3^n = 3^{n+1}$$

- Prove what you need for a small value (n=0) → $\sum_{i=0}^{0} 3^i = 3^0 < 3^{0+1}$

PROOF by INDUCTION

You want to prove something involving n

- Assume that it's true if you have n-1 — Inductive hypothesis
- Prove that this assumption helps — Inductive step
- Prove what you need for a small value — Base case

Prove: $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$ (if $n \geq 0$, $4^n - 1$ is divisible by 3)

Note: some sources define $\mathbb{N}$ as 1,2,3... instead of 0,1,2,3...

Prove: $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$ (if $n \geq 0$, $4^n - 1$ is divisible by 3)

Assume $3 \mid 4^{n-1} - 1$ and $n \geq 1$ because $n = 0$ will be base case

...?

Prove: $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$ (if $n \geq 0$, $4^n - 1$ is divisible by 3)

Assume $3 \mid 4^{n-1} - 1$ and $n \geq 1$ because $n=0$ will be base case

By definition, the assumption is: $\exists a \in \mathbb{Z}$ such that $3a = 4^{n-1} - 1$

...?

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$$4 \cdot \underline{3a} = 4 \cdot \underline{(4^{n-1} - 1)}$$

Prove: $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$ (if $n \geq 0$, $4^n - 1$ is divisible by 3)

Assume $3 \mid 4^{n-1} - 1$ and $n \geq 1$ because $n = 0$ will be base case

By definition, the assumption is: $\exists a \in \mathbb{Z}$ such that $3a = 4^{n-1} - 1$

↪ $4 \cdot 3a = 4 \cdot (4^{n-1} - 1) = 4^n - 4$

↪ ?

Prove: $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$ (if $n \geq 0$, $4^n - 1$ is divisible by 3)

Assume $3 \mid 4^{n-1} - 1$ and $n \geq 1$ because $n=0$ will be base case

By definition, the assumption is: $\exists a \in \mathbb{Z}$ such that $3a = 4^{n-1} - 1$

$\hookrightarrow$ $4 \cdot 3a = 4 \cdot (4^{n-1} - 1) = 4^n \underline{- 4}$

$\hookrightarrow$ $3 \cdot 4a \underline{+ 3} = 4^n \underline{- 1}$

$\hookrightarrow$?

Prove: $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$ (if $n \geq 0$, $4^n - 1$ is divisible by 3)

Assume $3 \mid 4^{n-1} - 1$ and $n \geq 1$ because $n=0$ will be base case

By definition, the assumption is: $\exists a \in \mathbb{Z}$ such that $3a = 4^{n-1} - 1$

$\hookrightarrow$ $4 \cdot 3a = 4 \cdot (4^{n-1} - 1) = 4^n - 4$

$\hookrightarrow$ $3 \cdot 4a + 3 = 4^n - 1$

$\hookrightarrow$ $3(4a+1) = 4^n - 1 \rightarrow$?

Prove: $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$ (if $n \geq 0$, $4^n - 1$ is divisible by 3)

Assume $3 \mid 4^{n-1} - 1$ and $n \geq 1$ because $n=0$ will be base case

By definition, the assumption is: $\exists a \in \mathbb{Z}$ such that $3a = 4^{n-1} - 1$

$\hookrightarrow$ $4 \cdot 3a = 4 \cdot (4^{n-1} - 1) = 4^n - 4$

$\hookrightarrow$ $3 \cdot 4a + 3 = 4^n - 1$

$\hookrightarrow$ $3\underbrace{(4a+1)}_{\in \mathbb{Z}} = 4^n - 1 \rightarrow$ By definition, $3 \mid 4^n - 1$ ✓

Prove: $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$ (if $n \geq 0$, $4^n - 1$ is divisible by 3)

Assume $3 \mid 4^{n-1} - 1$ and $n \geq 1$ because $n = 0$ will be base case

By definition, the assumption is: $\exists a \in \mathbb{Z}$ such that $3a = 4^{n-1} - 1$

$$\hookrightarrow 4 \cdot 3a = 4 \cdot (4^{n-1} - 1) = 4^n - 4$$

$$\hookrightarrow 3 \cdot 4a + 3 = 4^n - 1$$

$$\hookrightarrow 3\underbrace{(4a+1)}_{\in \mathbb{Z}} = 4^n - 1$$ → By definition, $3 \mid 4^n - 1$ ✓

Base case: $4^0 - 1 = 0$, so $3 \mid 0$ is true ✓

$\square$

Prove: $2^n > n^2$ for all $n \geqslant 5$

Prove: $2^n > n^2$ for all $n \geq 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geq 6$ because $n=5$ will be base case

Prove: $2^n > n^2$ for all $n \geq 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geq 6$ because $\underbrace{n=5 \text{ will be base case}}_{?}$

Prove: $2^n > n^2$ for all $n \geq 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geq 6$ because $n=5$ will be base case

$32 = 2^5 > 5^2 = 25$

Prove: $2^n > n^2$ for all $n \geq 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geq 6$ because $\underbrace{n=5 \text{ will be base case}}_{32 = 2^5 > 5^2 = 25}$

$$2^{n-1} > n^2 - 2n + 1$$

Prove: $2^n > n^2$ for all $n \geq 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geq 6$ because $\underbrace{n=5 \text{ will be base case}}_{32 = 2^5 > 5^2 = 25}$

$$2^{n-1} > n^2 - 2n + 1$$

$$2 \cdot 2^{n-1} > 2n^2 - 4n + 2$$

Prove: $2^n > n^2$ for all $n \geq 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geq 6$ because $\underbrace{n=5 \text{ will be base case}}_{32 = 2^5 > 5^2 = 25}$

$$2^{n-1} > n^2 - 2n + 1$$

$$2 \cdot 2^{n-1} > 2n^2 - 4n + 2$$

$$2^n > n^2 + (n^2 - 4n + 2)$$

Prove: $2^n > n^2$ for all $n \geq 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geq 6$ because $\underbrace{n=5 \text{ will be base case}}_{32 \,=\, 2^5 \,>\, 5^2 \,=\, 25}$

$$2^{n-1} > n^2 - 2n + 1$$

$$2 \cdot 2^{n-1} > 2n^2 - 4n + 2$$

$$2^n > n^2 + (n^2 - 4n + 2)$$

$$> n^2 + (n^2 - 4n - 5)$$

Prove: $2^n > n^2$ for all $n \geqslant 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geqslant 6$ because $\underbrace{n=5 \text{ will be base case}}_{32 = 2^5 > 5^2 = 25}$

$$2^{n-1} > n^2 - 2n + 1$$

$$2 \cdot 2^{n-1} > 2n^2 - 4n + 2$$

$$2^n > n^2 + (n^2 - 4n + 2)$$

$$> n^2 + (n^2 - 4n - 5)$$

$$= n^2 + (n-5)\cdot(n+1)$$

Prove: $2^n > n^2$ for all $n \geq 5$

Assume $2^{n-1} > (n-1)^2$ and $n \geq 6$ because $\underbrace{n=5 \text{ will be base case}}_{32 = 2^5 > 5^2 = 25}$

$$2^{n-1} > n^2 - 2n + 1$$

$$2 \cdot 2^{n-1} > 2n^2 - 4n + 2$$

$$2^n > n^2 + (n^2 - 4n + 2)$$

$$> n^2 + (n^2 - 4n - 5)$$

$$= n^2 + \underbrace{(n-5)\cdot(n+1)}_{> 0}$$

$$> n^2$$

$\square$

PROOF by INDUCTION

requiring more than one base case and
relying on more than one smaller instance

PROOF by INDUCTION requiring more than one base case and relying on more than one smaller instance

Fibonacci numbers: $F_0 = 1 \qquad F_1 = 1$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

PROOF by INDUCTION requiring more than one base case and relying on more than one smaller instance

Fibonacci numbers: $F_0 = 1 \qquad F_1 = 1$

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2$$

Prove: for all n, $F_n \leq 1.7^n$

PROOF by INDUCTION requiring more than one base case and relying on more than one smaller instance

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Fibonacci numbers: $F_0 = 1 \qquad F_1 = 1$

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2$$

Prove: for all n, $F_n \leq 1.7^n$

Base cases: $n=1$ & $n=2$, trivially true

PROOF by INDUCTION requiring more than one base case and relying on more than one smaller instance

Fibonacci numbers: $F_0 = 1$ $F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$

Prove: for all n, $F_n \leq 1.7^n$

Base cases: $n=1$ & $n=2$, trivially true

Hypothesis: for $n \geqslant 2$ assume $F_{n-1} \leq 1.7^{n-1}$, $F_{n-2} \leq 1.7^{n-2}$

PROOF by INDUCTION requiring more than one base case and relying on more than one smaller instance

Fibonacci numbers: $F_0 = 1 \qquad F_1 = 1$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

Prove: for all n, $F_n \leq 1.7^n$

Base cases: $n=1$ & $n=2$, trivially true

Hypothesis: for $n \geq 2$ assume $F_{n-1} \leq 1.7^{n-1}$, $F_{n-2} \leq 1.7^{n-2}$

$F_n \leq 1.7^{n-1} + 1.7^{n-2}$ by definition and substitution

PROOF by INDUCTION requiring more than one base case and relying on more than one smaller instance

Fibonacci numbers: $F_0 = 1 \qquad F_1 = 1$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

Prove: for all n, $F_n \leq 1.7^n$

Base cases: $n=1$ & $n=2$, trivially true

Hypothesis: for $n \geq 2$ assume $F_{n-1} \leq 1.7^{n-1}$, $F_{n-2} \leq 1.7^{n-2}$

$$F_n \leq 1.7^{n-1} + 1.7^{n-2} = (1.7 + 1) \cdot 1.7^{n-2}$$

PROOF by INDUCTION requiring more than one base case and relying on more than one smaller instance

Fibonacci numbers: $F_0 = 1 \qquad F_1 = 1$

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2$$

Prove: for all n, $F_n \leq 1.7^n$

Base cases: $n=1$ & $n=2$, trivially true

Hypothesis: for $n \geq 2$ assume $F_{n-1} \leq 1.7^{n-1}$, $F_{n-2} \leq 1.7^{n-2}$

$$F_n \leq 1.7^{n-1} + 1.7^{n-2} = (1.7+1)\cdot 1.7^{n-2} < (2\cdot 1.7)\cdot 1.7^{n-2}$$

PROOF by INDUCTION requiring more than one base case and relying on more than one smaller instance

Fibonacci numbers: $F_0 = 1 \qquad F_1 = 1$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

Prove: for all n, $F_n \leq 1.7^n$

Base cases: $n=1$ & $n=2$, trivially true

Hypothesis: for $n \geq 2$ assume $F_{n-1} \leq 1.7^{n-1}$, $F_{n-2} \leq 1.7^{n-2}$

$$F_n \leq 1.7^{n-1} + 1.7^{n-2} = (1.7+1) \cdot 1.7^{n-2} < (2 \cdot 1.7) \cdot 1.7^{n-2} = 1.7^n$$

$\square$

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

0
1
1
2
3
5
8
13
21
34
55
89
⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases: $\begin{cases} (n=0) : F_0 \ \& \ F_3 \text{ are both even} : 0 \ \& \ 2 \\ (n=1) : F_1 \ \& \ F_4 \text{ are both odd} : 1 \ \& \ 3 \end{cases}$

0
1
1
2
3
5
8
13
21
34
55
89
⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases: $\begin{cases} (n=0) : F_0 \;\&\; F_3 \text{ are both even} : 0 \;\&\; 2 \\ (n=1) : F_1 \;\&\; F_4 \text{ are both odd} : 1 \;\&\; 3 \end{cases}$

Assume $S(n-1)$ and $S(n-2)$ are true

0
1
1
2
3
5
8
13
21
34
55
89
⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases: $\begin{cases} (n=0) : F_0 \ \& \ F_3 \text{ are both even} : 0 \ \& \ 2 \\ (n=1) : F_1 \ \& \ F_4 \text{ are both odd} : 1 \ \& \ 3 \end{cases}$

Assume $S(n-1)$ and $S(n-2)$ are true

F_{n-1} & F_{n+2} : both even or both odd

0
1
1
2
3
5
8
13
21
34
55
89
⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases: $(n=0)$: F_0 & F_3 are both even : 0 & 2

$(n=1)$: F_1 & F_4 are both odd : 1 & 3

Assume $S(n-1)$ and $\underline{S(n-2)}$ are true

F_{n-1} & F_{n+2} : both even or both odd

F_{n-2} & F_{n+1} : both even or both odd

0
1
1
2
3
5
8
13
21
34
55
89
⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases:

$(n=0)$: F_0 & F_3 are both even : 0 & 2

$(n=1)$: F_1 & F_4 are both odd : 1 & 3

Assume $S(n-1)$ and $S(n-2)$ are true

F_{n-1} & F_{n+2} : both even or both odd

F_{n-2} & F_{n+1} : both even or both odd

By definition, if F_n even then F_{n-1}, F_{n-2} both even or both odd.

of Fibonacci numbers (and integer addition)

0
1
1
2
3
5
8
13
21
34
55
89
⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases:
- $(n=0)$: F_0 & F_3 are both even : 0 & 2
- $(n=1)$: F_1 & F_4 are both odd : 1 & 3

Assume $S(n-1)$ and $S(n-2)$ are true

F_{n-1} & F_{n+2} : both even or both odd

F_{n-2} & F_{n+1} : both even or both odd

By definition, if F_n even then F_{n-1}, F_{n-2} both even or both odd.

If both even then ...?

0
1
1
2
3
5
8
13
21
34
55
89
⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases:
$(n=0)$: F_0 & F_3 are both even : 0 & 2
$(n=1)$: F_1 & F_4 are both odd : 1 & 3

Assume $S(n-1)$ and $S(n-2)$ are true

F_{n-1} & F_{n+2} : both even or both odd

F_{n-2} & F_{n+1} : both even or both odd

By definition, if F_n even then F_{n-1}, F_{n-2} both even or both odd.

If both even then by hypothesis ...?

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases: $\begin{cases} (n=0) : & F_0 \ \& \ F_3 \text{ are both even} : \ 0 \ \& \ 2 \\ (n=1) : & F_1 \ \& \ F_4 \text{ are both odd} : \ 1 \ \& \ 3 \end{cases}$

Assume $S(n-1)$ and $S(n-2)$ are true

$\boxed{F_{n-1} \ \& \ F_{n+2} : \text{both even or both odd}}$

$F_{n-2} \ \& \ F_{n+1}$: both even or both odd

By definition, if F_n even then F_{n-1}, F_{n-2} both even or both odd.

If both even then by hypothesis F_{n+2} even

0
1
1
2
3
5
8
13
21
34
55
89
⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

Base cases:
- $(n=0)$: F_0 & F_3 are both even : 0 & 2
- $(n=1)$: F_1 & F_4 are both odd : 1 & 3

Assume $S(n-1)$ and $S(n-2)$ are true

F_{n-1} & F_{n+2} : both even or both odd

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By definition, if F_n even then F_{n-1}, F_{n-2} both even or both odd.

If both even then by hypothesis F_{n+2}, F_{n+1} even

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⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even

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by definition

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ⋮

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If both odd then ...?

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0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ⋮

Prove statement $S(n)$: F_n is even IFF F_{n+3} is even 0

1

Base cases:
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3

Assume $S(n-1)$ and $S(n-2)$ are true 5

F_{n-1} & F_{n+2} : both even or both odd 8

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21

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89

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Similar proof if F_n is odd.

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- Sometimes you will see the inductive step assuming that Claim(n) is known and using it to show Claim$(n+1)$.

I tend to assume Claim$(n-1)$ and prove Claim(n).

This is a matter of style. Both are ok.

Tips:

- The base case is not always "the smallest number"
  - Sometimes a claim is true only for larger values. $2^n > n^5$
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  make sure you have enough base cases

For all $n \geq 0$, $2^n = 1$ (?!)

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(fact, for all n)

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Needed base case for $n=1$

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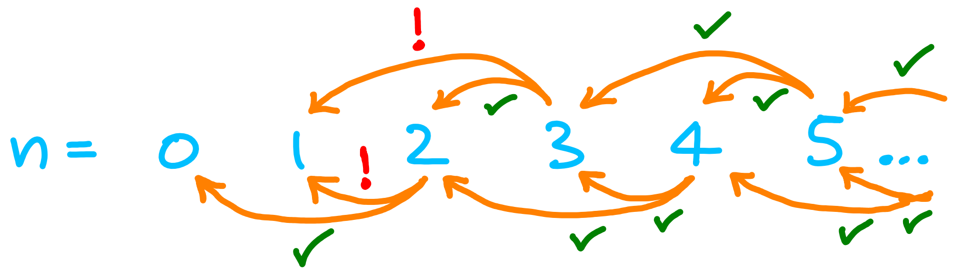
Needed base case for $n=1$

but $2^1 \neq 1$



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FYI - Regular induction and strong induction are actually equivalent as is proof by smallest counterexample.

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(possibly more than two of them)

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Proved, without knowing exactly what n relied on.

Prove: for all n, $\displaystyle\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$

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Prove: for all n, $\displaystyle\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$

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$$\sum_{x=1}^{n} x = n + \sum_{x=1}^{n-1} x = n + \frac{(n-1)\cdot n}{2} = \frac{2n + n^2 - n}{2}$$

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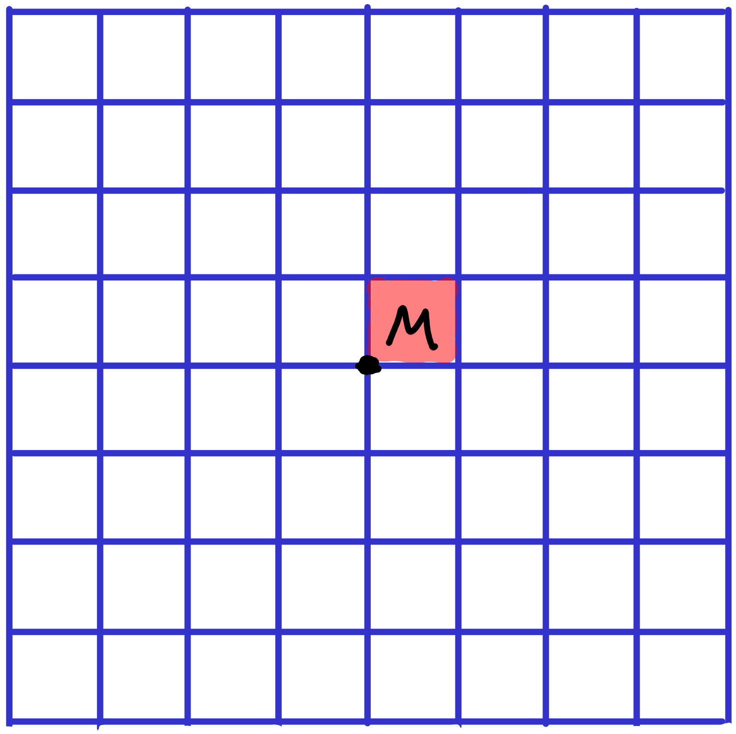
$\square$

Let $S(n)$ be a grid of $2^n \times 2^n$ squares with one square in the "middle", marked.



8×8

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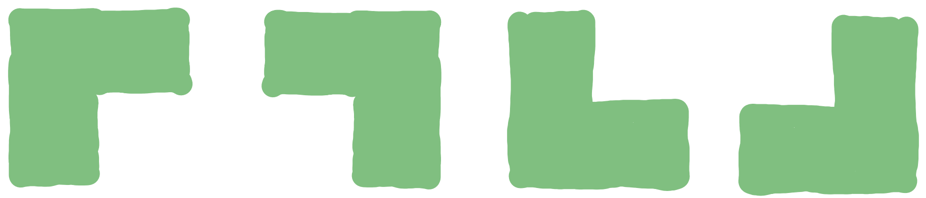
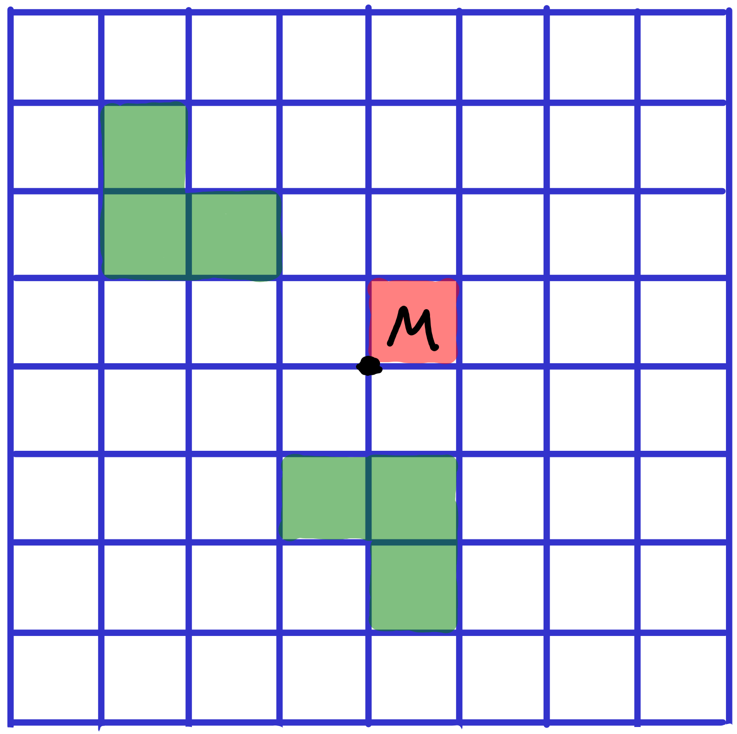
You have L-shaped tiles that you can rotate and place on the grid.





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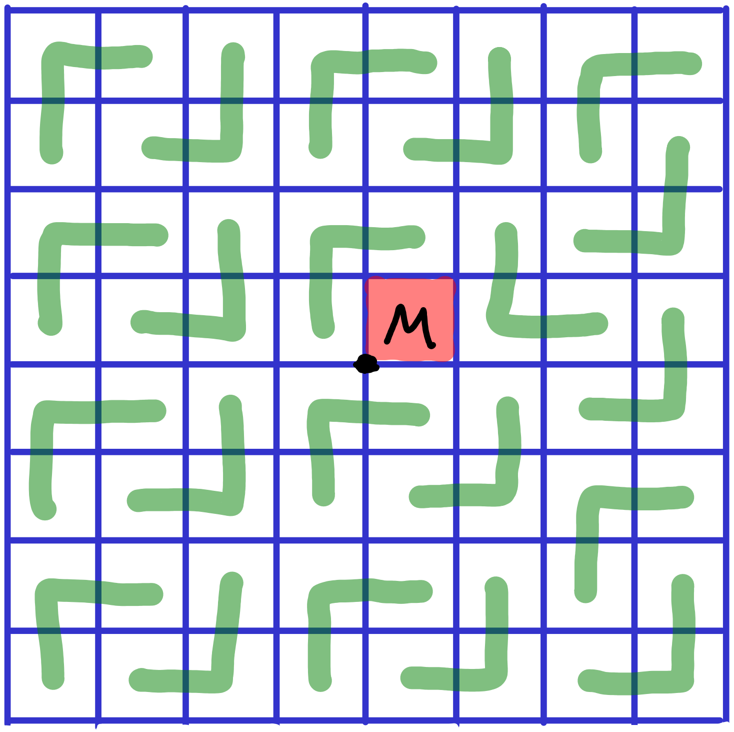
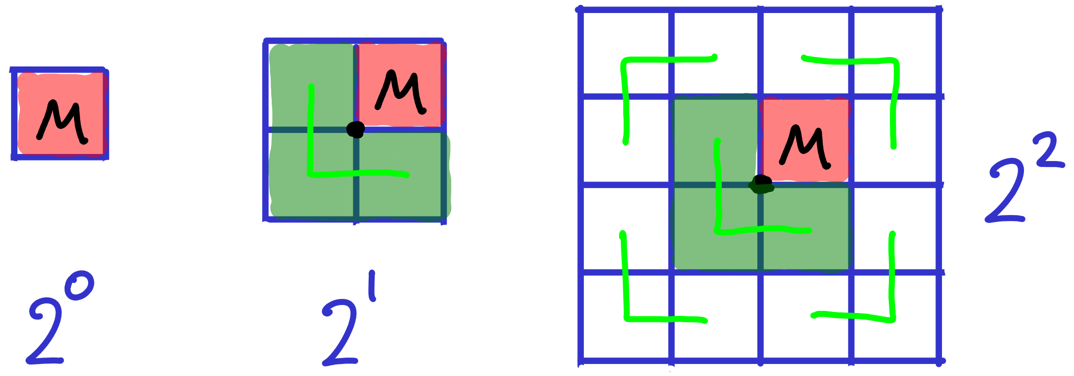
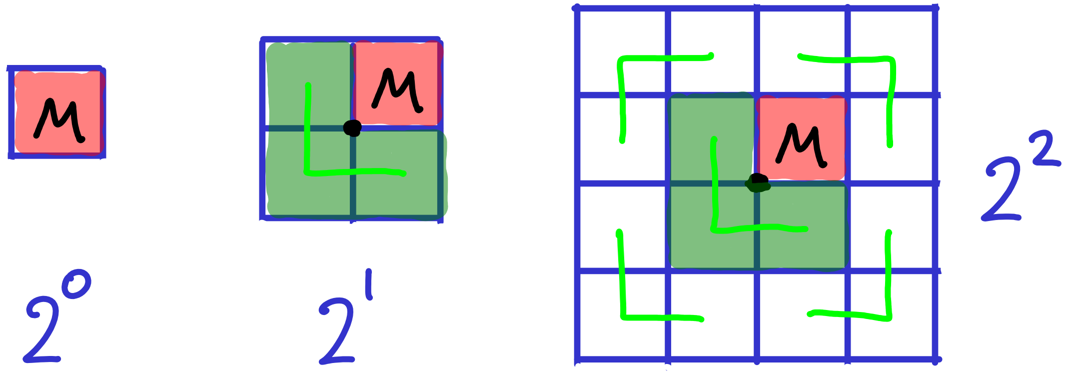


You have L-shaped tiles that you can rotate and place on the grid.

Prove that you can cover the grid with tiles, except for M.



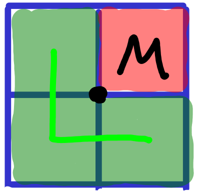
Works for small n (base case)



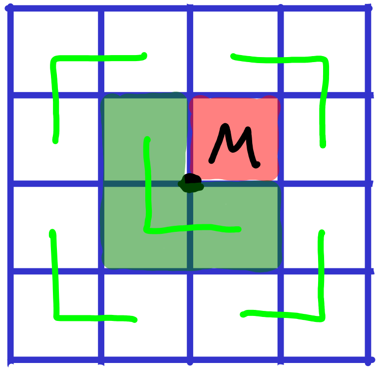
Works for small n (base case)

Hypothesis: $2^{n-1} \times 2^{n-1}$ can be tiled $(n \geq 1)$

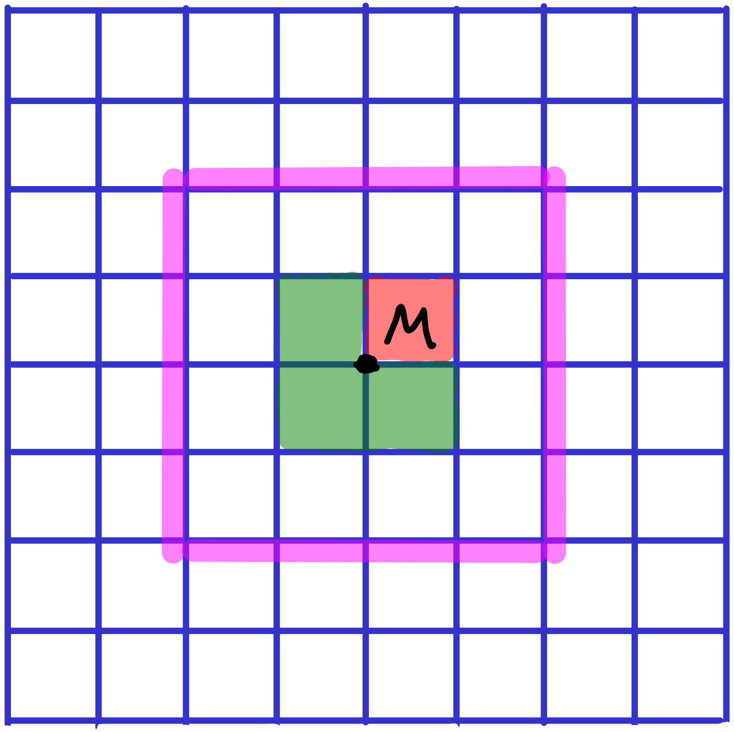
2^0



2^1



2^2



2^3

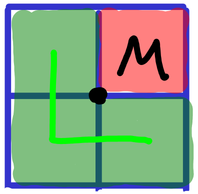
Works for small n (base case)

Hypothesis : $2^{n-1} \times 2^{n-1}$ can be tiled $(n \geq 1)$

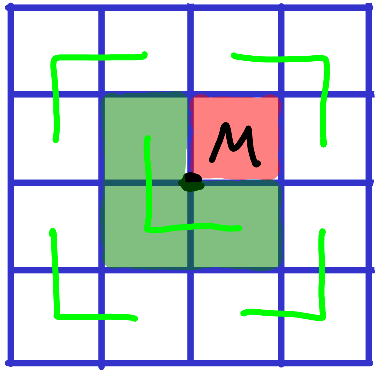
But there is no way to use this!

Actual hypothesis = $2^{n-1} \times 2^{n-1}$ grid with M in the middle can be tiled

2^0

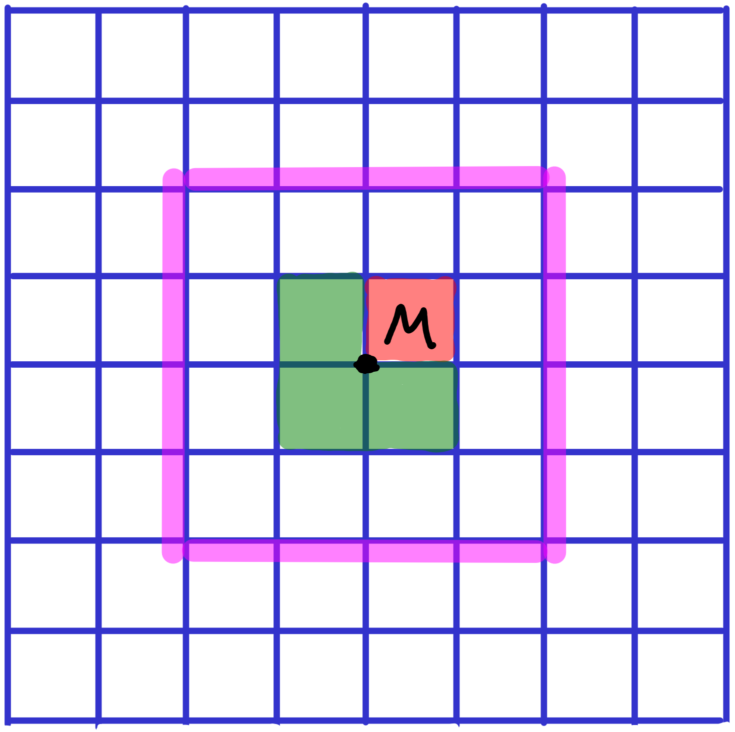


2^1



2^2

2^3



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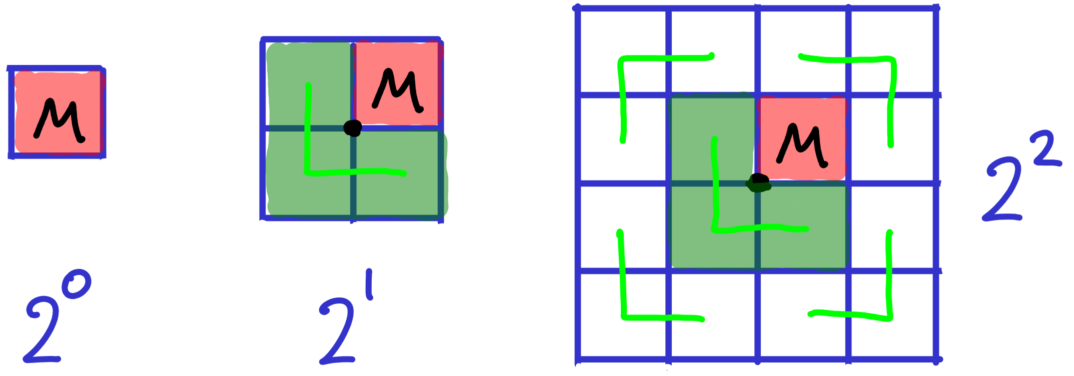
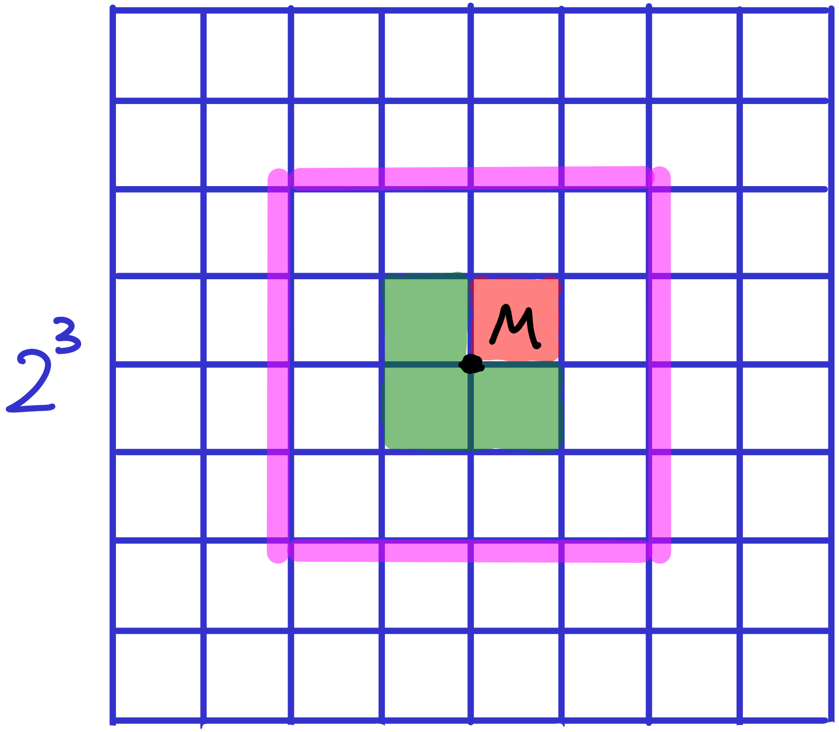
Hypothesis must match claim





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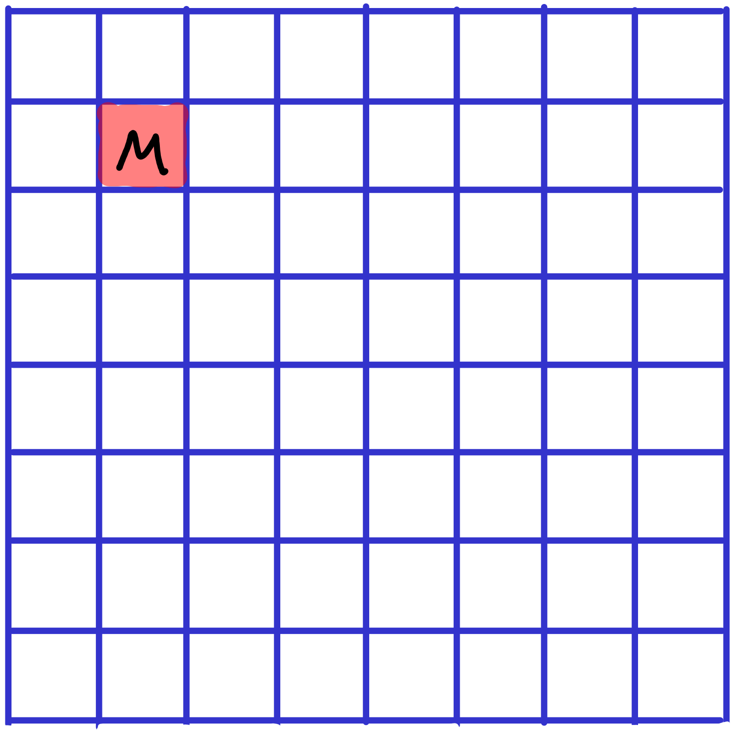
Solution: make problem harder...

Let $S(n)$ be a grid of $2^n \times 2^n$ squares with one square ~~in the "middle"~~ somewhere, marked.

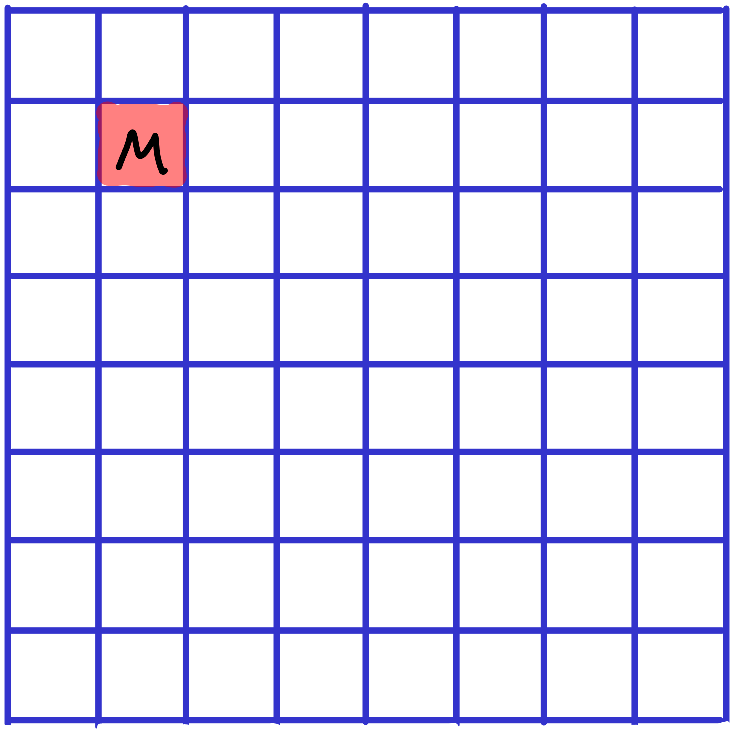


The new problem is harder.

We are given less information.

Let $S(n)$ be a grid of $2^n \times 2^n$ squares with one square ~~in the "middle"~~ somewhere, marked.

Prove: grid can be covered except for M.



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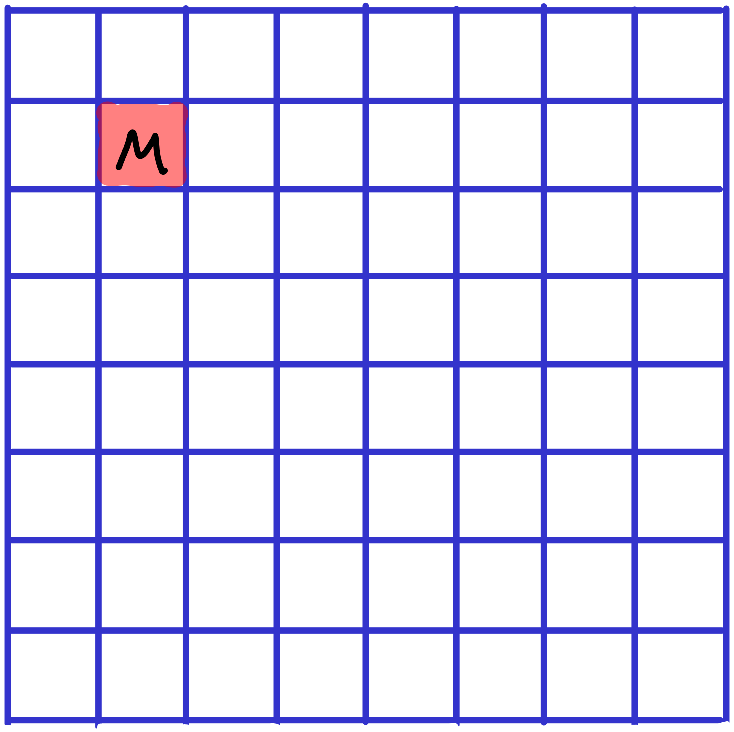
with one square in ~~the "middle"~~, marked.

somewhere



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Hypothesis: $2^{n-1} \times 2^{n-1}$ can be tiled $(n \geq 1)$

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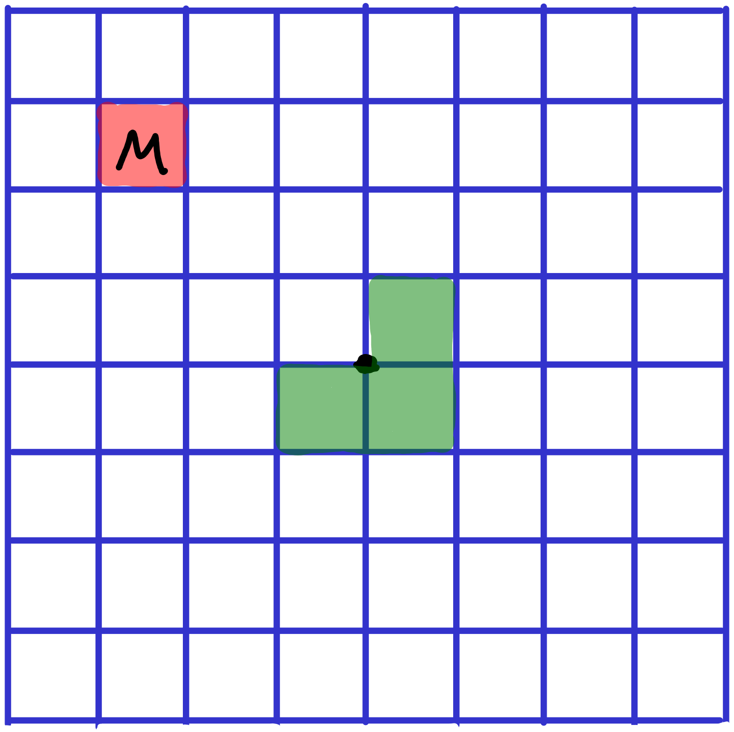

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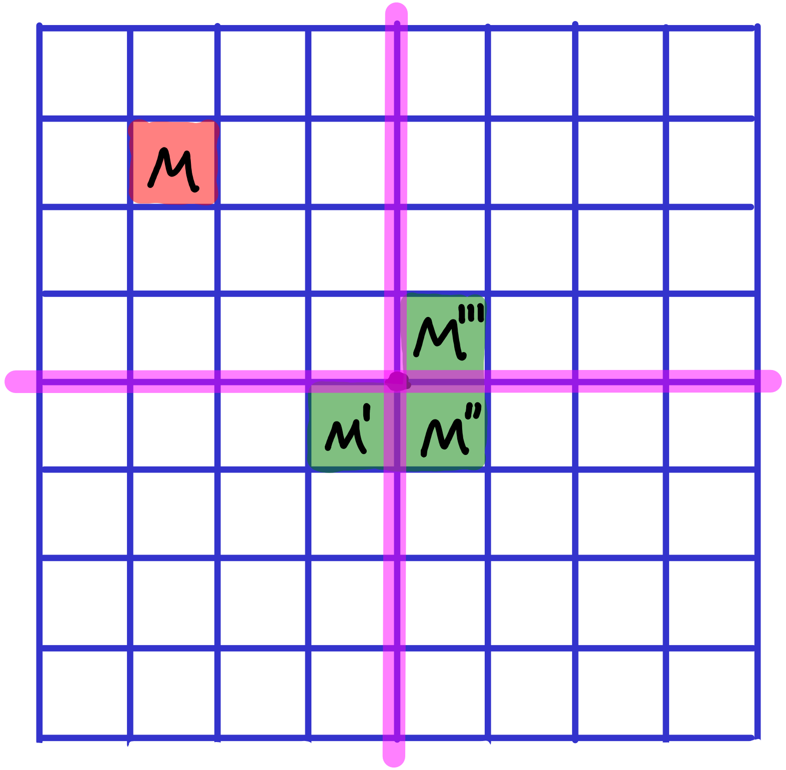

Prove: grid can be covered except for M.

Base case: easy

Hypothesis: $2^{n-1} \times 2^{n-1}$ can be tiled $(n \geq 1)$

Place tile in the middle, avoiding quadrant with M.

By hypothesis, each quadrant can be tiled.

$\square$

Why was it easier to solve a harder problem?

The inductive hypothesis became more powerful.

Let's see another example...

Prove: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2} \leqslant 2$ for all $n \geqslant 1$

Prove: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} \leqslant 2$ for all $n \geqslant 1$

Base case: $n=1$: $1 \leqslant 2$ ✓

Prove: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2} \leqslant 2$ for all $n \geqslant 1$

Base case: $n=1$: $1 \leqslant 2$ ✓

Assume $\sum_{i=1}^{n-1} \frac{1}{i^2} \leqslant 2$ for $n \geqslant 2$

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Assume $\sum_{i=1}^{n-1} \frac{1}{i^2} \leqslant 2$ for $n \geqslant 2$

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Base case: $n=1$: $1 \leqslant 2$ ✓

Assume $\sum_{i=1}^{n-1} \frac{1}{i^2} \leqslant 2$ for $n \geqslant 2$

$$\sum_{i=1}^{n} \frac{1}{i^2} = \frac{1}{n^2} + \sum_{i=1}^{n-1} \frac{1}{i^2} \leqslant \frac{1}{n^2} + 2$$

Prove: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} \leqslant 2$ for all $n \geqslant 1$

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FAIL

so let's prove something stronger...

Prove: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$ for all $n \geq 1$

in fact $<$ for $n \geq 2$

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$$\sum_{i=1}^{n} \frac{1}{i^2} = \frac{1}{n^2}+\sum_{i=1}^{n-1} \frac{1}{i^2}$$

Prove: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all $n \geq 1$

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Prove: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$ for all $n \geq 1$

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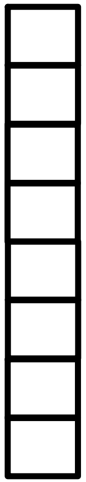
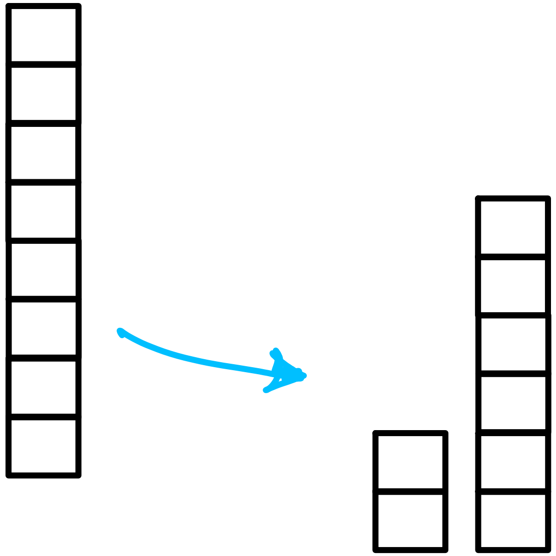
$\square$

You have a stack of n boxes.



You have a stack of n boxes.

One move: split a stack into 2 new stacks of size a & b.



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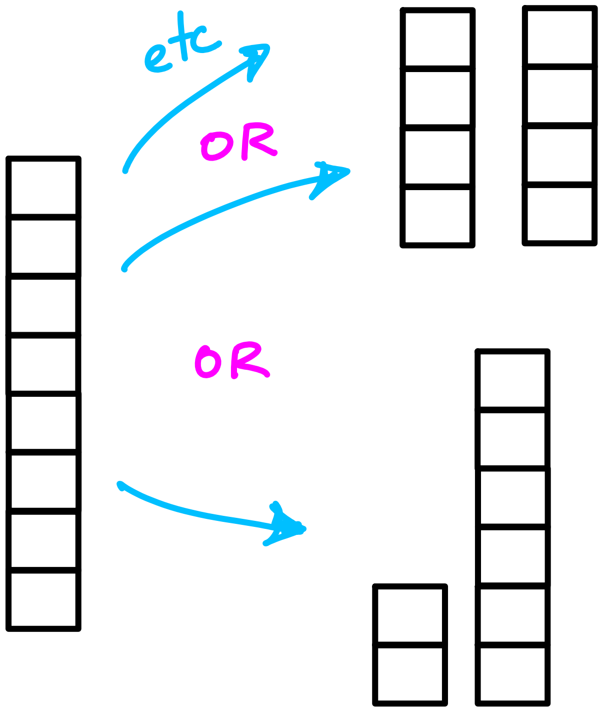

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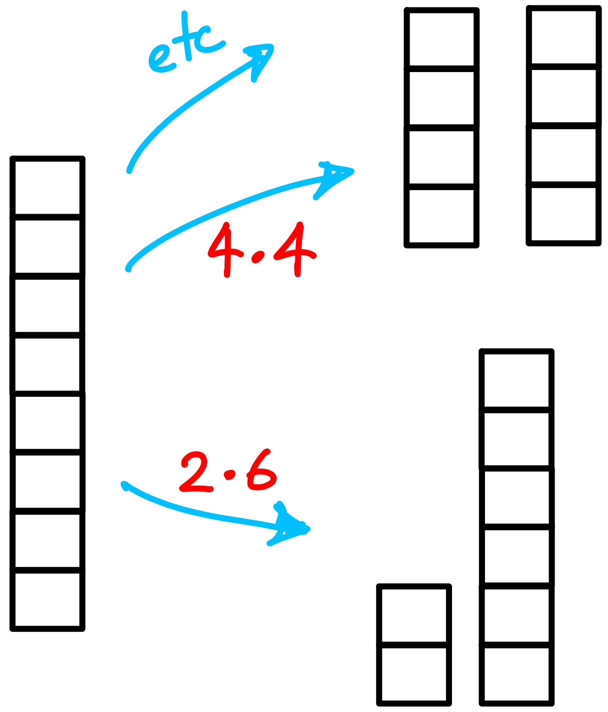

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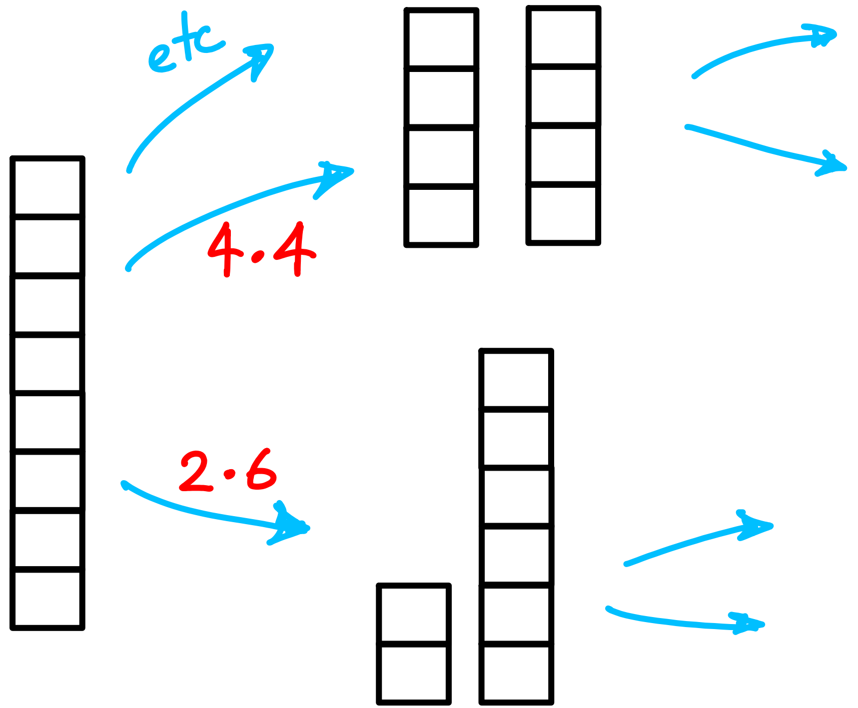
Do this until all stacks have size 1.

Try to maximize reward.



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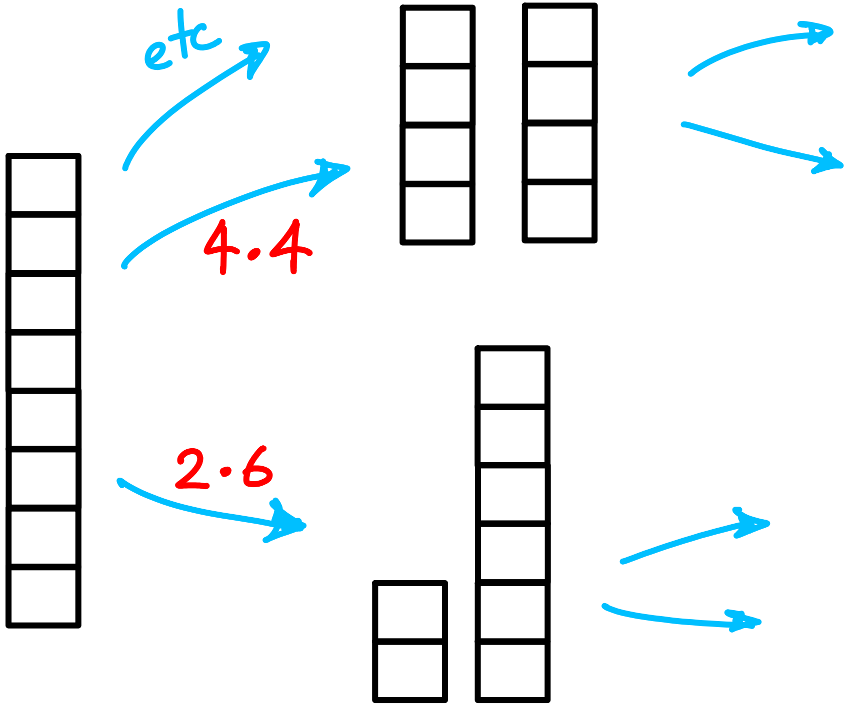

Try to balance a, b always?

Product is maximized when equal

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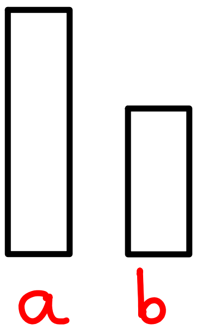
Consider stack of $\underset{a+b}{\underset{\downarrow}{n}}$ boxes. Let 1st move produce stacks $a, b < n$.



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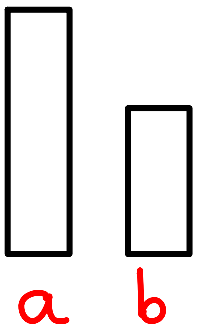

By hypothesis, future reward $= \frac{a(a-1)}{2} + \frac{b(b-1)}{2}$

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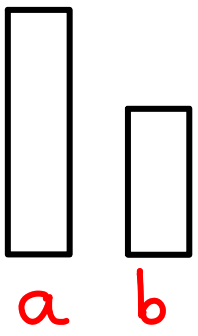

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$$\text{Total} = \frac{2ab + a^2 - a + b^2 - b}{2}$$

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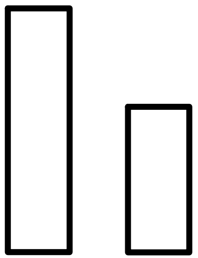

$a \quad b$

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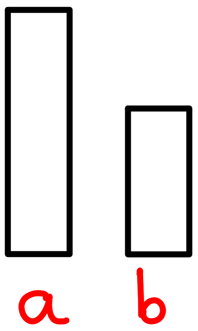

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$$\text{Total} = \frac{2ab + a^2 - a + b^2 - b}{2} = \frac{(a+b)^2 - (a+b)}{2} = \frac{n^2 - n}{2} = \frac{n(n-1)}{2} \quad \square$$

$$\text{Let} \begin{cases} T(n) = T\left(\frac{n}{2}\right) + n & \text{for all } n \geq 2 \text{ such that } n = 2^d,\ d \in \mathbb{N} \\ T(1) = 1 \end{cases}$$

i.e., $n = 2, 4, 8, 16, \ldots$

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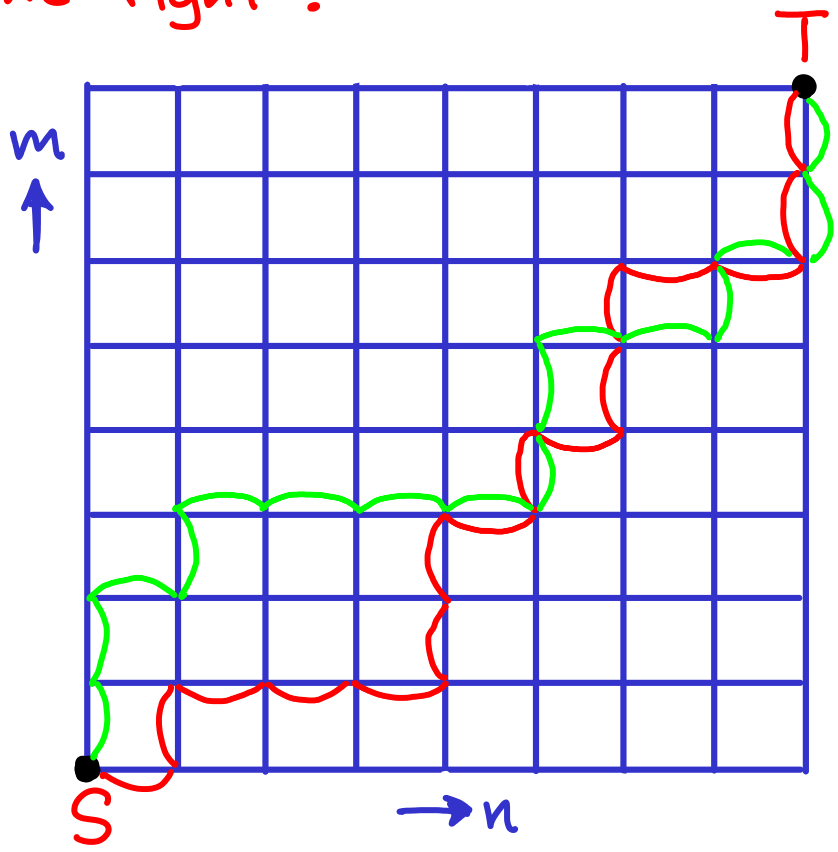
strong induction,
didn't need $T(n-1)$

So $T(n) = (n-1) + n = 2n - 1$

$\square$

How many ways are there to get from $S=(0,0)$ to $T=(n,m)$ if you always take a step up or to the right?



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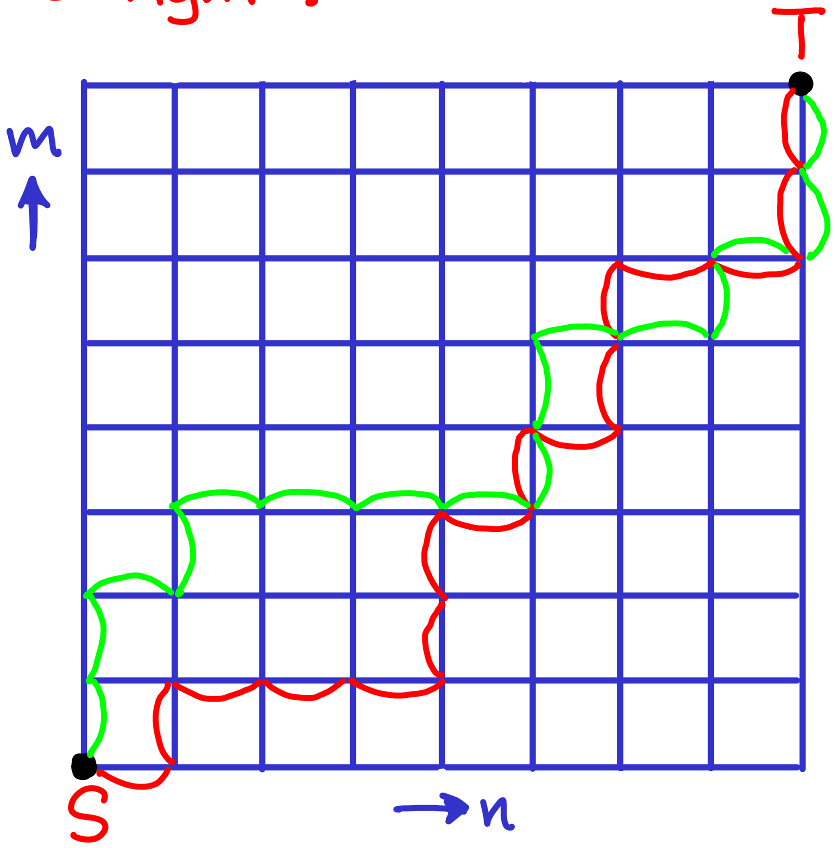
Prove: $W(n,m) = \dfrac{(n+m)!}{n!\,m!}$



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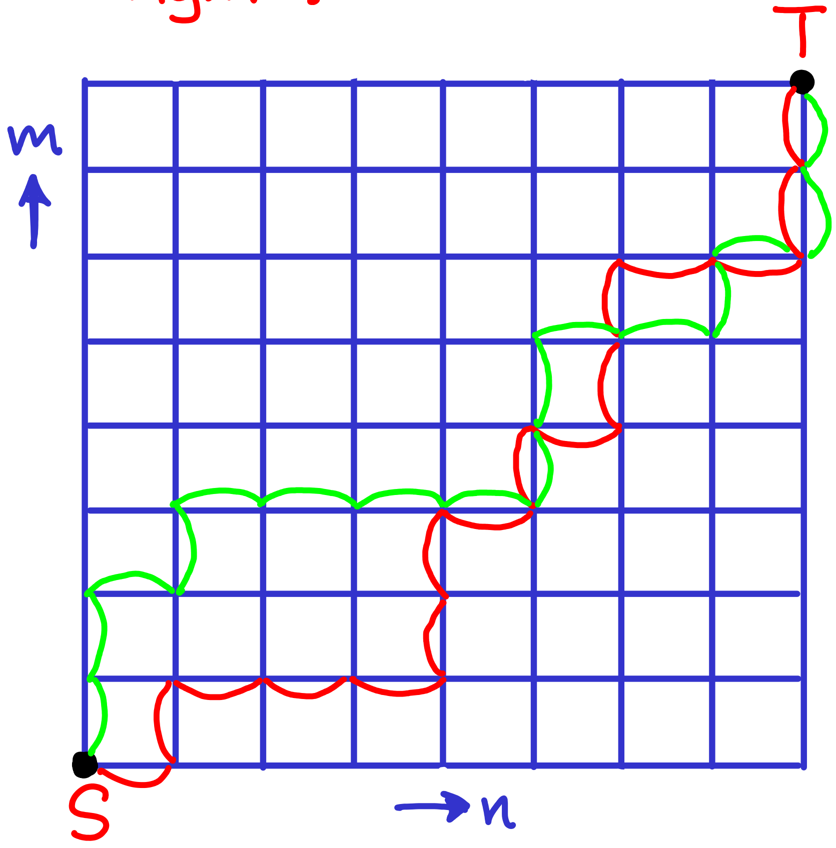

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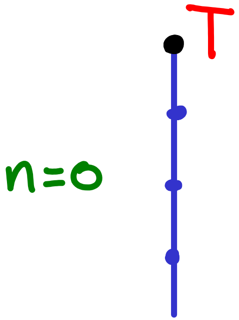
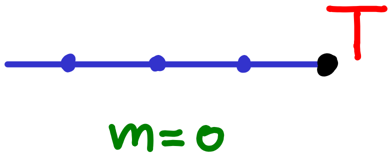
Base cases: all grids where $m=0$ or $n=0$

$$W(n,0) = W(0,m) = 1 \quad \text{(fact)}$$





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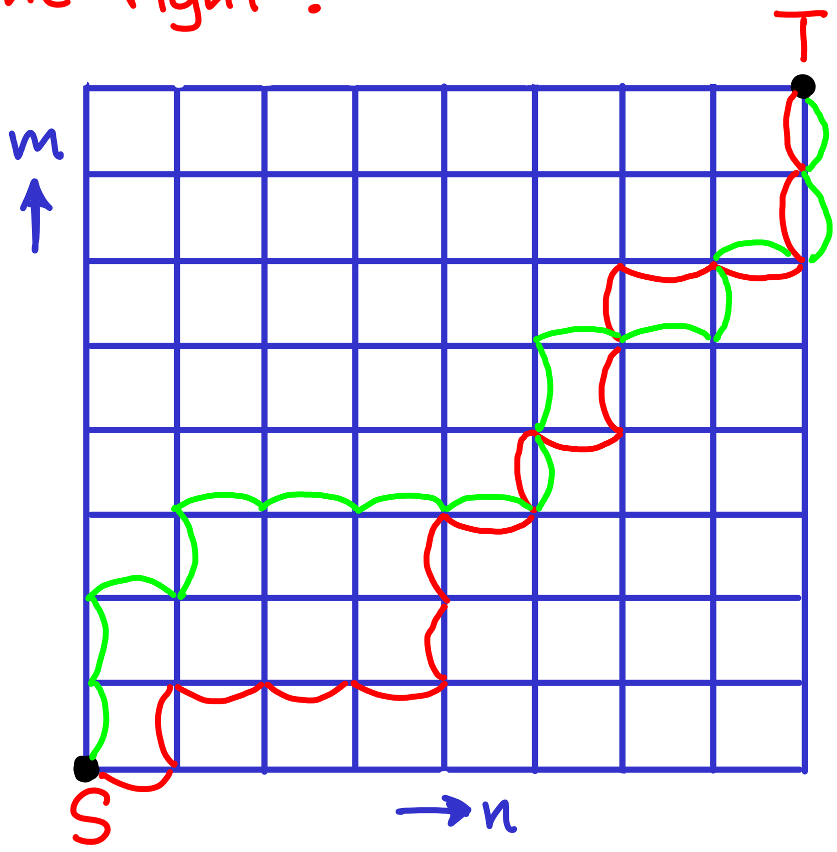


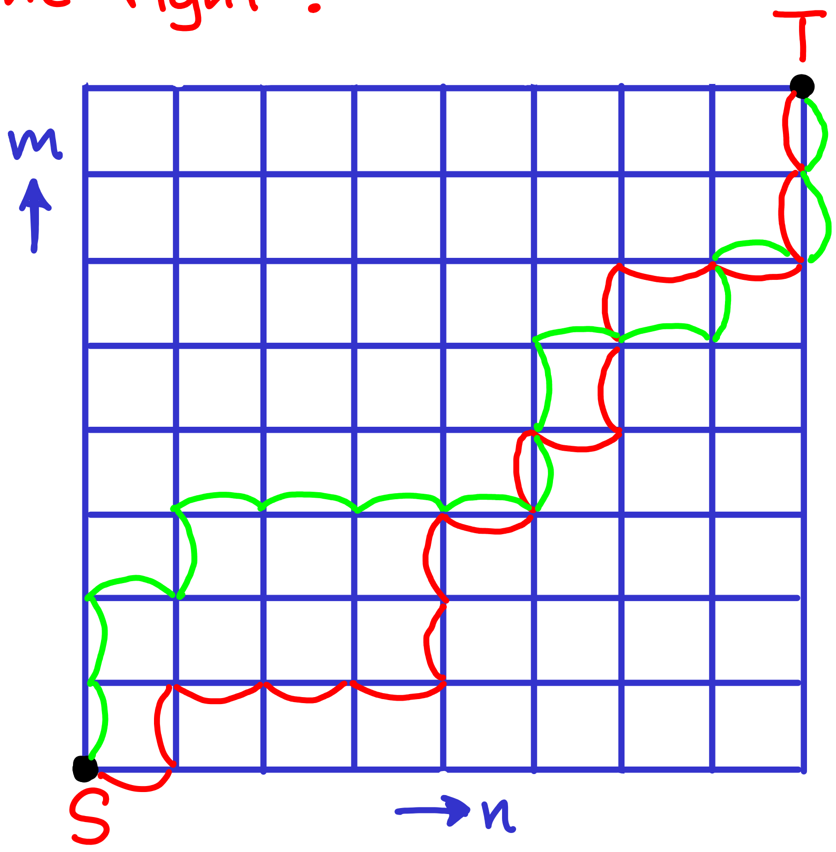
Base cases: all grids where $m=0$ or $n=0$

$$W(n,0) = W(0,m) = 1 \quad \text{(fact)}$$

If $m=0$, $\frac{(n+m)!}{n!\,m!} = \frac{(n+0)!}{n!\,0!} = 1$ ✓

(similar if $n=0$)

How many ways are there to get from $S=(0,0)$ to $T=(n,m)$ if you always take a step up or to the right?



Prove: $W(n,m) = \dfrac{(n+m)!}{n!\,m!}$

Base cases: all grids where $m=0$ or $n=0$

$$W(n,0) = W(0,m) = 1 \quad \text{(fact)}$$

If $m=0$, $\dfrac{(n+m)!}{n!\,m!} = \dfrac{(n+0)!}{n!\,0!} = 1$ ✓

Hypothesis: if $1 \leq x \leq n$, $1 \leq y \leq m$, $x+y < n+m$,

(i.e., if remaining grid is smaller) then $W(x,y) = \dfrac{(x+y)!}{x!\,y!}$

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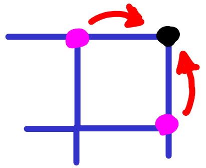

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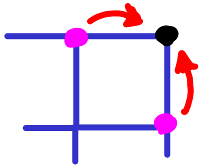

If not base case then $n, m \geq 1$.

By hypothesis

$$W(n-1,m) = \frac{((n-1)+m)!}{(n-1)!\,m!}$$

ok because $(n-1)+m < n+m$

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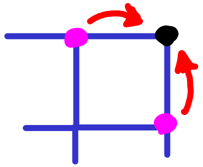

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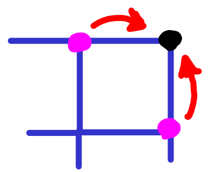

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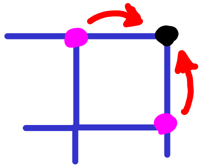

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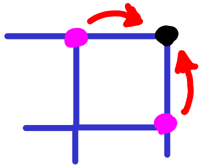

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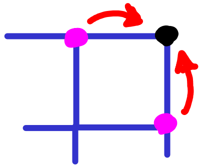

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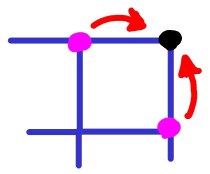

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- induction with 2 variables
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- how a hypothesis can have creative conditions,

  e.g., $x+y < n+m$

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What went wrong? → Argument fails for $n=2$

Be careful of general statements