It is assumed that this is known:

$$a^n = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n \text{ times}}$$

Also, the properties on the right are used in one example.

$$2^a \cdot 2^b = 2^{a+b}$$

$$2^{ab} = (2^a)^b = 2^{a \cdot b}$$

$$\sqrt{x} = x^{1/2}$$

$$\sqrt{x^a} = (x^a)^{1/2} = x^{a/2}$$

$$\log a^b = \log(a^b) = b \cdot \log a$$

A proposition is a statement that is either true or false.

Examples: $\boxed{2+3=5 \quad T}$ $\boxed{1+1=3 \quad F}$

Not proposition: Will I get an A? I will get an A.

Why not? It's a question. It will eventually be T or F.

"is" vs "will eventually be"

This is quite subtle. Do you have free will?

A proposition is a statement that is either true or false.

Examples: $\boxed{2+3=5 \quad T}$ $\boxed{1+1=3 \quad F}$

Not proposition: Will I get an A? I will get an A.

Why not? It's a question. It will eventually be T or F.

Proposition: If I get 100% on my exams, I'll get an A.

yes, assuming no asteroids strike, I didn't cheat, etc.

(let's keep things simple)

A proposition is a statement that is either always true or always false.

Examples: $\boxed{2+3=5 \quad T}$ $\boxed{1+1=3 \quad F}$

Not proposition: Will I get an A? I will get an A.

Why not? It's a question. It will eventually be T or F.

Proposition: If I get 100% on my exams, I'll get an A.

Not proposition: There are clouds in the sky.

Why not? At this moment it's T or F, but it may vary.

Proposition: If it's raining then there are clouds in the sky.

Proposition ? : This statement is $\underbrace{\text{either true or false.}}_{\text{true}}$

...but what does it mean to say "this statement is true" ?

The statement is somewhat meaningless.

Proposition ? : This statement is not a proposition.

ENJOY!

A conjecture is a proposition for which someone thinks there is a specific answer.

Ideally not just a guess.

In 1769, Euler conjectured: There are no integers $a,b,c,d > 0$ such that $a^4 + b^4 + c^4 = d^4$.

This is a proposition. Nobody knew if it's T or F for 218 years but that is not the same as

Not proposition: "I will get an A."

Why not? It will eventually be T or F.

The conjecture was shown false:

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

(counterexample)

In 1769 it was already false.

Notation

$\exists$: exists

$\forall$: for all

$\in$: "in", "member of"

$\mathbb{Z}$: set of integers

There are no integers $a,b,c,d > 0$ such that $a^4 + b^4 + c^4 = d^4$.

$$\underbrace{\nexists}_{\text{"does not exist"}} a,b,c,d \in \underbrace{\mathbb{Z}^+}_{\text{positive integers}}.\ a^4 + b^4 + c^4 = d^4$$

think of this as "such that"

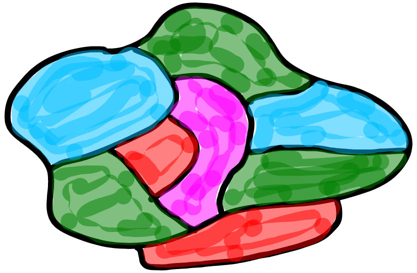
Rephrase: for all integers $a,b,c,d > 0$, $a^4 + b^4 + c^4 \neq d^4$.

$$\forall a,b,c,d \in \mathbb{Z}^+.\ a^4 + b^4 + c^4 \neq d^4$$

conditions . result

Theorem: an important proposition that has been proved true.

4-color theorem



Fermat's last "theorem"

1630 → ... → ... → 1994

(claim) (Conj.) (Thm.)

proved for $n < 4 \cdot 10^6$

$$x^n + y^n \neq z^n$$

for all positive integers x, y, z and for all integers $n > 2$

Goldbach's conjecture: Every integer >2 is the sum of two primes.

1742 ...

True up to at least 10^{18}.

Lemma: basically a mini theorem. ← Subjective

Typically something to be used as part of the proof of a more important result.

Corollary: a result that follows immediately from a theorem.

Axiom: something that you accept as true. (no proof)

$1+1=2$. MJ = GOAT.

→ Not the same as Assumption

When proving $x^n + y^n \neq z^n$ you can't assume x, y, z are even but you can use the properties of addition & powers, etc.

What is the easiest possible proof that a proposition P is true?

If P is an axiom, done.

Suppose P is not an axiom.

If P is directly implied by some axiom A, great.

implies

A is true. $A \to P$, so P is true.

In general, A need not be an axiom, it just needs to be true.

Prove P: prove A, then prove $A \to P$

Prove P: prove A, then prove $A \to P$

IF A is true AND $A \to P$ is true THEN P is true

Shorthand: If A and $A \to P$ then P

$(A \text{ and } A \to P) \to P$: "modus ponens"

$$\big(\underbrace{A \text{ and } A \to B}\text{ and } B \to C \text{ and } C \to P\big) \to P$$

$$\big(\underbrace{B \text{ and } B \to C}\text{ and } C \to P\big) \to P$$

$$(C \text{ and } C \to P) \to P \quad \checkmark$$

How do you prove $P \to Q$?

How do you prove IF P THEN Q ?

Sometimes all it takes is to break it down into simpler statements.

- If $0 \le x \le 2$, $x \in \mathbb{Z}$, then $-x^3 + 4x + 1 > 0$.

If x is equal to 0, 1 or 2 then $-x^3 + 4x + 1 > 0$.

$$\left\{\begin{array}{ll} & \text{If } x = 0 \text{ then } -x^3 + 4x + 1 > 0 \\ \text{AND} & \text{If } x = 1 \text{ then } -x^3 + 4x + 1 > 0 \\ \text{AND} & \text{If } x = 2 \text{ then } -x^3 + 4x + 1 > 0 \end{array}\right.$$

Case analysis

How do you prove $P \to Q$?

How do you prove IF P THEN Q ?

Sometimes all it takes is to break it down into simpler statements.

- If $0 \le x \le 2$, ~~$x \in \mathbb{Z}$~~, then $-x^3 + 4x + 1 > 0$. ?

If $0 \le x \le 2$, then $x \cdot (-x^2 + 4) + 1 > 0$.

If $0 \le x \le 2$, then $x \cdot (2^2 - x^2) + 1 > 0$.

If $0 \le x \le 2$, then $\underbrace{\underbrace{x}_{\geq 0} \cdot \underbrace{(2 - x)}_{\geq 0} \cdot \underbrace{(2 + x)}_{\geq 0}}_{\geq 0} + 1 > 0$. ✓

Same proof, different style

- If $\overbrace{0 \le x \le 2}^{P}$, then $\overbrace{-x^3 + 4x + 1 > 0}^{Q}$. ?

If $0 \le x \le 2$, then $x \geq 0$, $2 - x \geq 0$, $2 + x \geq 0$. $P \to A$

then $x \cdot (2 - x) \cdot (2 + x) \geq 0$. $A \to B$

then $x \cdot (2 - x) \cdot (2 + x) + 1 > 0$. $B \to C$

then $x \cdot (2^2 - x^2) + 1 > 0$. $C \to D$

then $-x^3 + 4x + 1 > 0$. $D \to Q$

✓

Sometimes it works well to just simplify $P \to Q$ until it's easy.

for $x, y, z > 0$, $x^5 \cdot (x \cdot z)^4 \cdot 2^y \cdot \log x^3 = 3x^9 \cdot \log x \cdot 2^{y/2} \cdot 2^{y/2} \cdot \sqrt{z^8}$

for $x, y, z > 0$, $x^9 \cdot z^4 \cdot 2^y \cdot 3\log x = 3x^9 \cdot \log x \cdot 2^{y/2+y/2} \cdot z^{8/2}$

for $x, y, z > 0$, $\cancel{x^9} \cdot \cancel{z^4} \cdot \cancel{2^y} \cdot \cancel{3\log x} = \cancel{3x^9 \cdot \log x} \cdot \cancel{2^{y/2+y/2}} \cdot \cancel{z^{8/2}}$

for $x, y, z > 0$, $1 = 1$ ✓

Sometimes $P \to A$, $A \to B$, $B \to C$, $C \to Q$ will be more clear or shorter.

You might even combine both.

- Mention what type of proof you are using.
- Define anything that isn't already given. & don't redefine
- Don't assume – especially don't assume what you're proving.
  - ↳ exceptions: case analysis, induction, proof by contradiction...

- Justify every step
- Don't skip steps
- Avoid saying "obviously"

unless it is truly obvious :)
(somewhat subjective)

IFF : "if and only iff"

P IFF Q is equivalent to $\underbrace{P \to Q \text{ AND } Q \to P}_{P \leftrightarrow Q}$

P is true IF Q is true AND P is true ONLY IF Q is true

if Q then P

if P then Q

$Q \to P$ AND $P \to Q$

For more intuition, see truth tables & contrapositive.