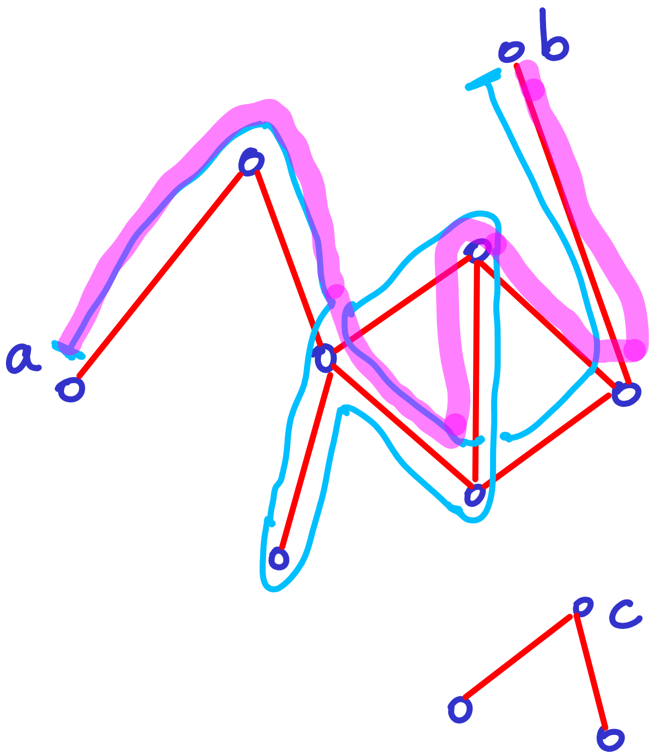
GRAPH CONNECTIVITY



A **walk** is a sequence of vertices $v_i, v_{i+1}, v_{i+2}, \dots, v_k$

s.t. every v_j, v_{j+1} is an edge $(i \leq j < k)$

We can walk from a to b, but not from a to c.

A **path** is a walk with distinct vertices

GRAPH CONNECTIVITY

A vertex x is connected to a vertex y (in G)
if there is a path from x to y

could say "walk"

If there is an x-y walk in G then there is an x-y path in G.

↳ Pick the shortest x-y walk that is not a path.

It must have a repeated vertex, u } $x \ldots u \underbrace{\ldots}_{\text{remove}} u \ldots y$

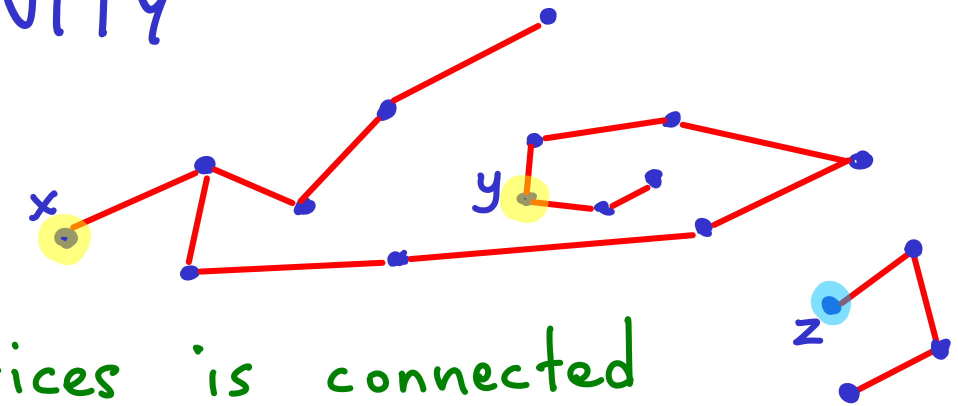
CONTRADICTION

□

GRAPH CONNECTIVITY

A graph is connected

if every pair of vertices is connected



If vertex x is connected to vertex y

then they are in the same component.

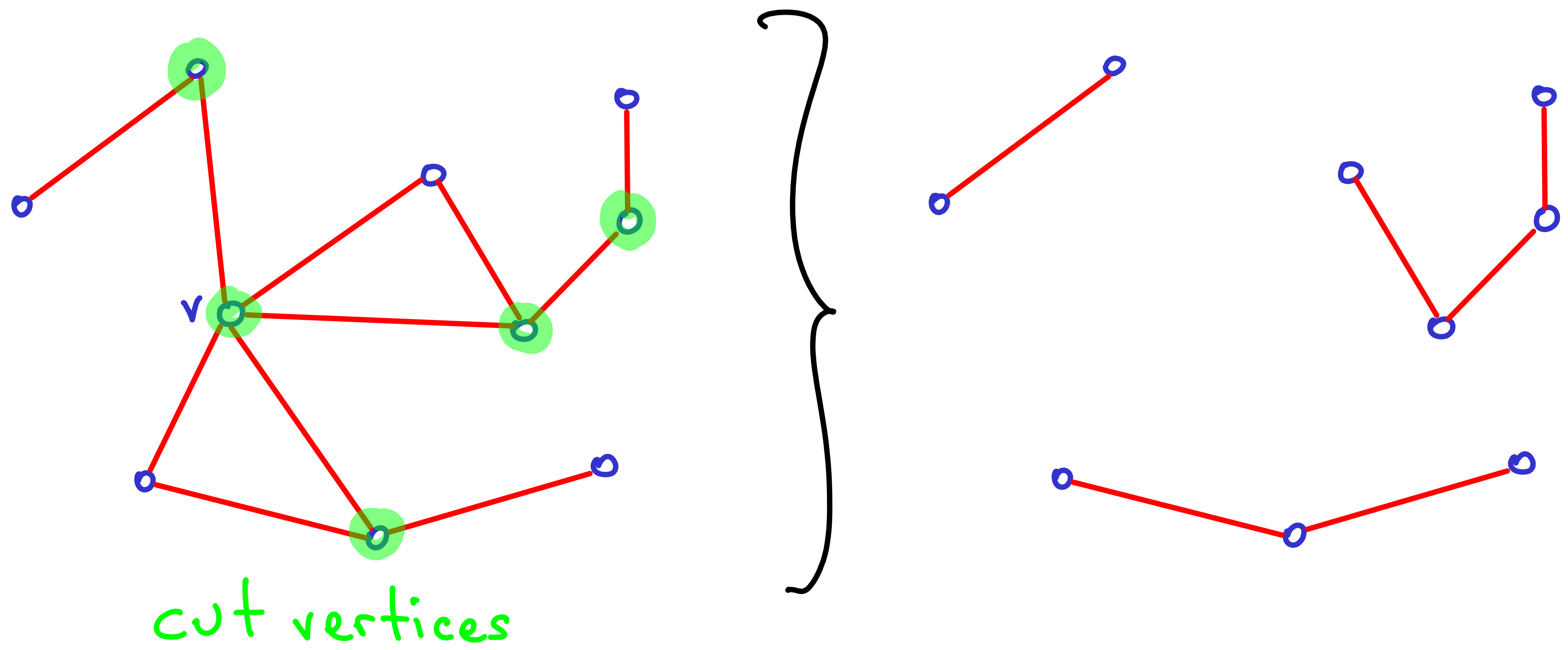
If x & y are in the same component

but x & z are not

then y & z are not.

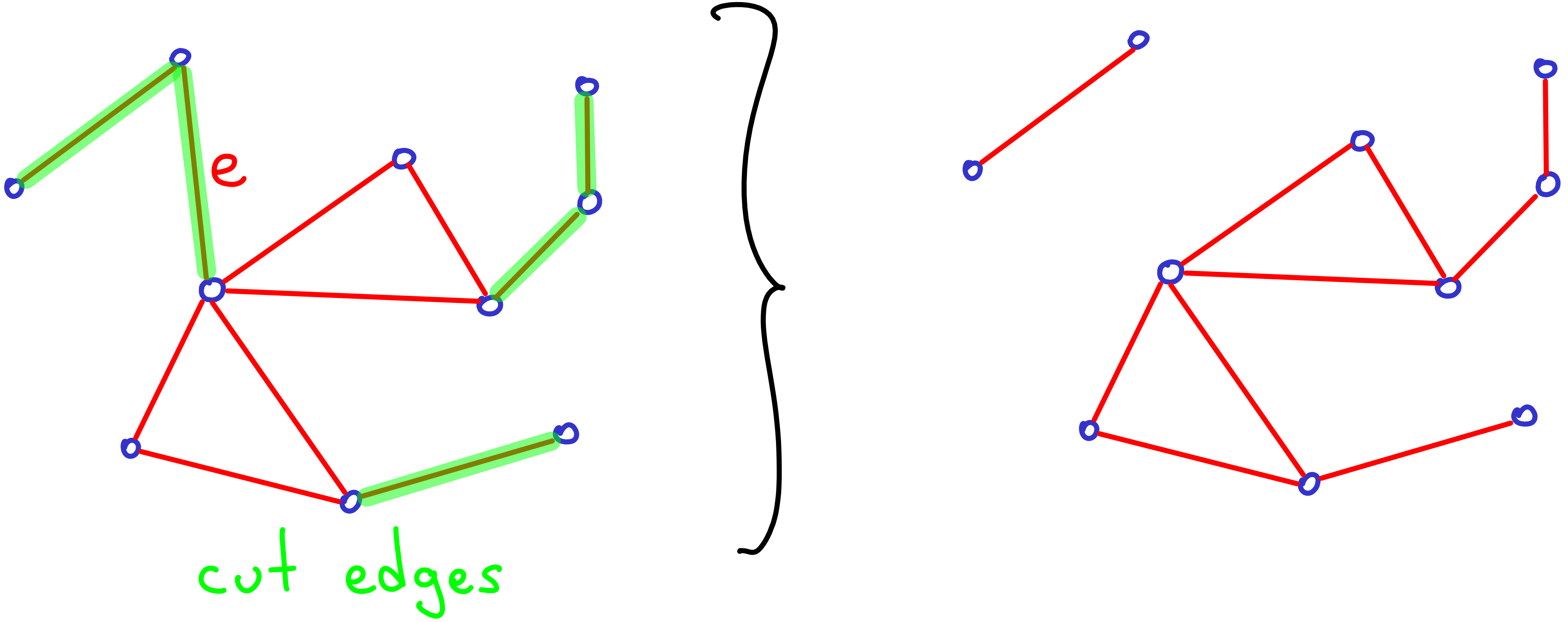
Given G, remove a vertex: G-v

If G-v has more components than G, then v is a cut vertex.



Given G, remove an edge : G - e

If G-e has more components than G, then e is a cut edge.



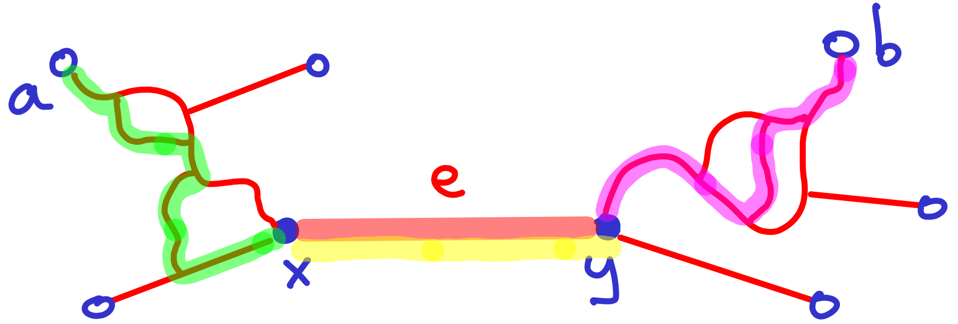
...cont'd: if e is a cut edge then $\exists\, a, b$ s.t.

a is not connected to b in $G - e$.

... but they are connected in G.

So all paths from a to b in G use e,

i.e., $\exists$ path $a \cdots xy \cdots b$ where $e = \overline{xy}$.



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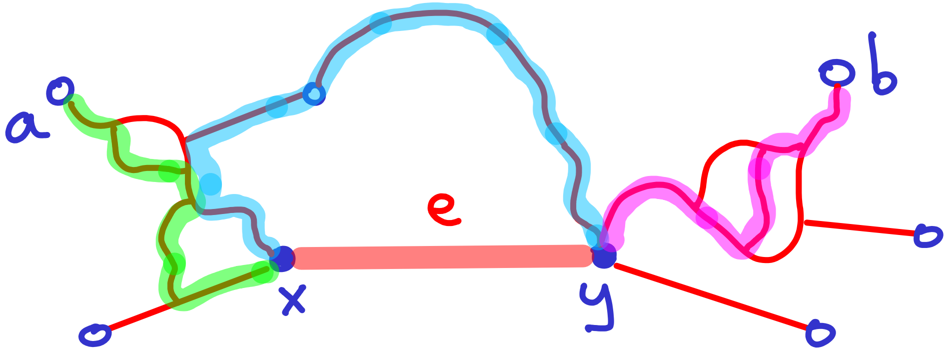


But if e is on a cycle then we can walk from a to x, walk from x to y, without using e, and walk from y to b.

CONTRADICTION

$\square$

observation

if e is a cut edge then $\exists\ a, b$ s.t. a is not connected to b in $G - e$.

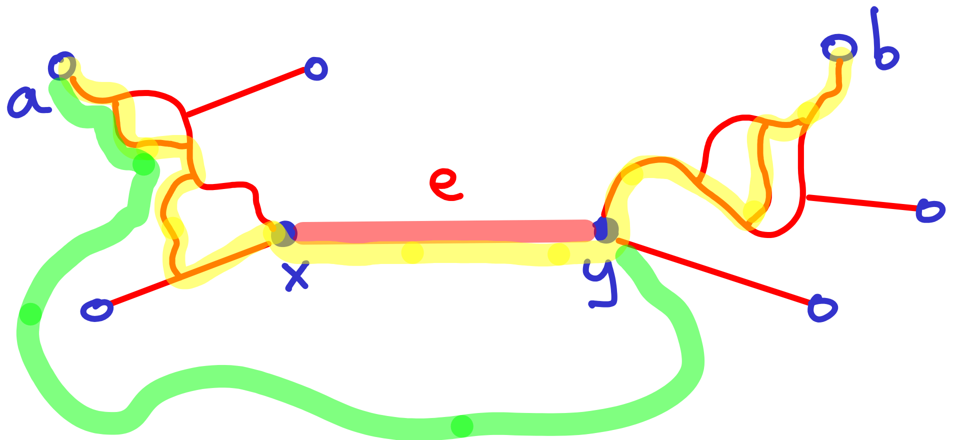
So all paths from a to b in G use e,

i.e., $\boxed{\exists}$ path $a \cdots xy \cdots b$ where $e = \overline{xy}$.

↳ all paths from a to b must traverse $a \cdots xy \cdots b$

not $a \cdots yx \cdots b$

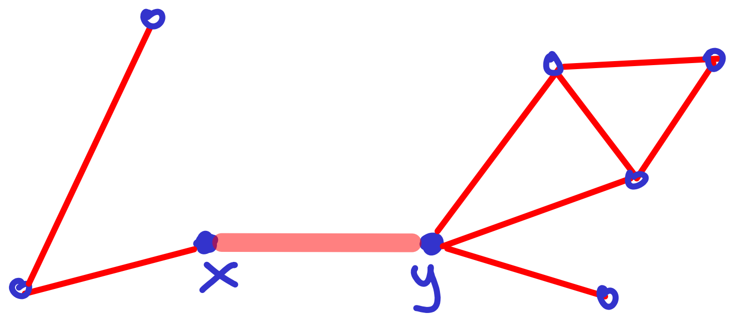
would create a cycle



□

Claim: Removing a cut edge $e = (x, y)$ increases the number of components by 1.

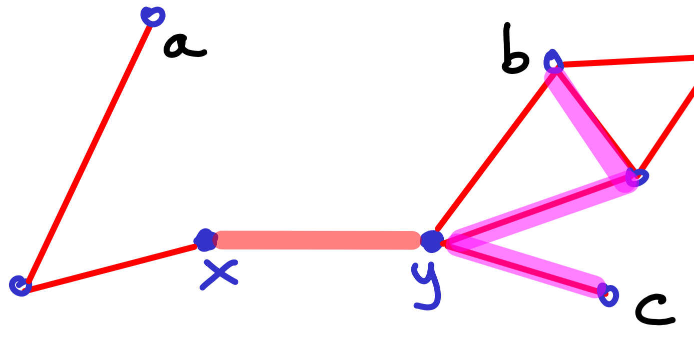
* e can only affect the component it's in. So focus on connected graphs.



By definition of cut edge, we get at least 1 new component.

The question is: why not > 1 ?

Claim: Removing a cut edge $e = (x, y)$ increases the number of components by 1.



* e can only affect the component it's in.
So focus on connected graphs.

Proof by contradiction.

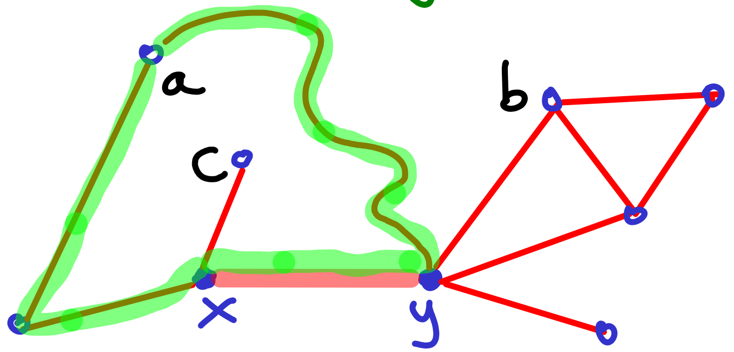
Suppose $G - e$ has ≥ 3 components. $\exists\, a, b, c$ in different components.

In G, all paths $a \to b$ use e } wlog $a \to x \to y \to b$

all paths $a \to c$ use e { if $a \to x \to y \to c$

$\Rightarrow$ b & c in same component

Claim: Removing a cut edge $e = (x, y)$ increases the number of components by 1.



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all paths $a \to c$ use e { if $a \to x \to y \to c$

$\Rightarrow$ b & c in same component

if $a \to y \to x \to c$

$\Rightarrow$ e: not a cut edge

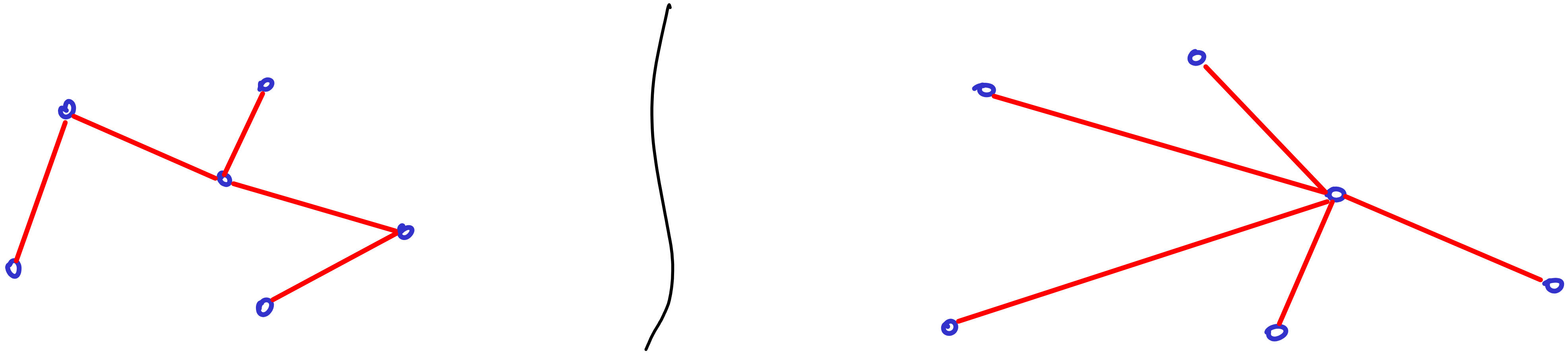
$\square$

Recap

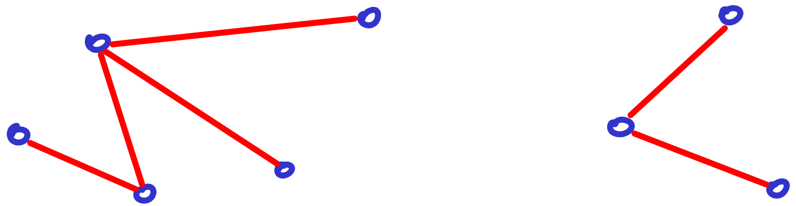
1) A cut edge can't be on a cycle.

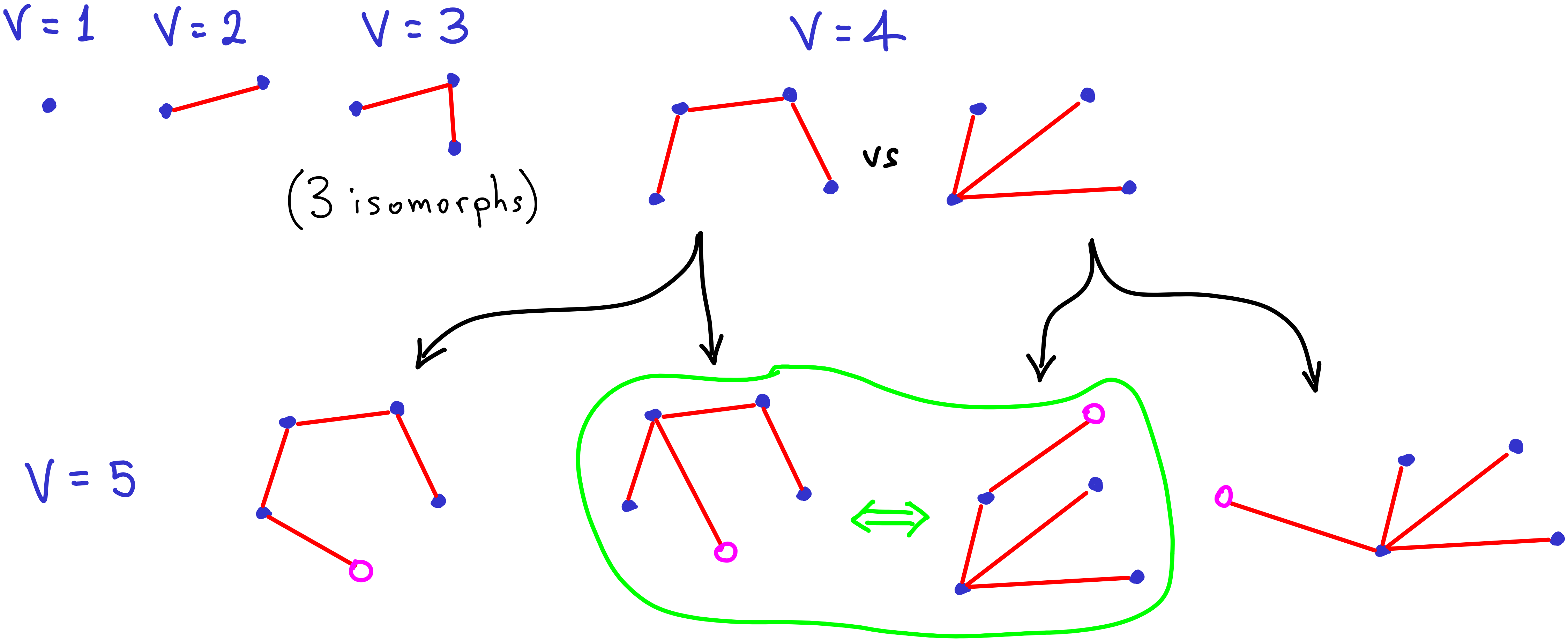
2) Removing a cut edge $e = (x, y)$ increases the number of components by 1.

TREES : CONNECTED ACYCLIC GRAPHS



FORESTS : ACYCLIC GRAPHS (collections of trees)




V = 1
V = 2
V = 3
(3 isomorphs)
V = 4
vs
V = 5

tree $\iff$ there is a unique path between every pair of vertices

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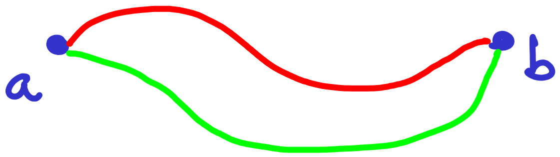
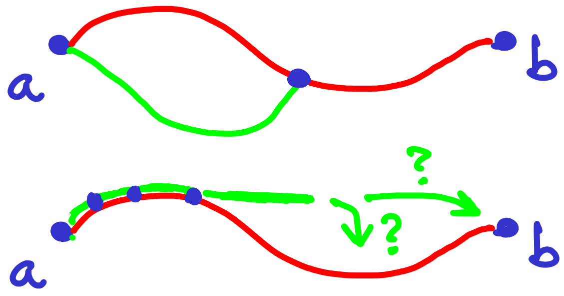
$\Rightarrow$ • for any vertices a, b : a path exists (trees are connected)

• suppose ≥ 2 paths.





↳ cycle : contradiction of tree: acyclic

$\Leftarrow$ • if for every 2 vertices a path exists, then graph is connected

• if any 2 vertices are on a unique path, they are not on a cycle.

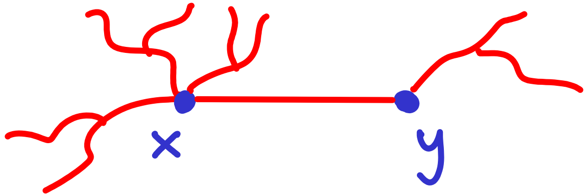
↳ acyclic

$\square$

For any connected graph,

tree $\iff$ every edge is a cut edge

$\Rightarrow$



} for any $\overline{xy}$ } tree $\Rightarrow$ unique path from x to y

$\Rightarrow$ $\overline{xy}$: cut edge

$\Leftarrow$ suppose graph $\neq$ tree.

Then it has a cycle.

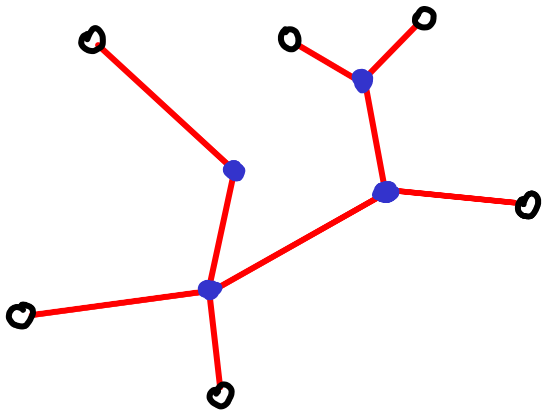
We have proved: A cut edge can't be on a cycle.

not every edge is a cut edge. (CONTRADICTION)

$\square$

LEAVES : vertices of degree 1

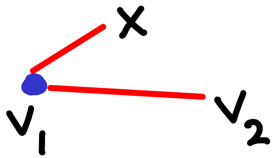
If $V \geqslant 2$, then T has $\geqslant 2$ leaves



Consider longest path in T. $v_1 \dots v_k$

$(k \geqslant 2)$

If $v_1 \neq$ leaf, then



$x \neq v_i$ (not on path)

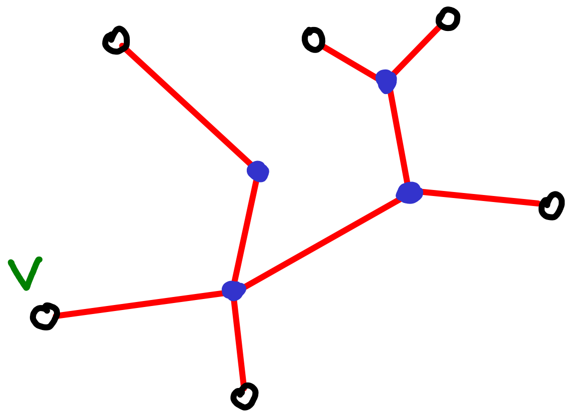
($x = v_i$ would create cycle)

Then $x v_1 \dots v_k$: longer path : CONTRADICTION

So v_1 & v_k : leaves $\square$

If v is a leaf in tree T, then $T-v$ is a tree

- removing v doesn't create cycles.
- removing v doesn't disconnect.
  ($v \neq$ cut vertex $\rightarrow T-v$ is connected)



↳ if v were a cut vertex, then $\exists\, a,b$ $(a \neq v,\ b \neq v)$ s.t.
any path $a \rightarrow b$ must use v.
But v is a dead end:
can't be part of such a path

in fact, "the only path" (T: unique paths)

□

If v is a leaf in tree T, then $T-v$ is a tree

This allows us to use induction for:

if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$ ✓

pf: Base case: $n=2$  trivial

Hypothesis: for $2 \leq k < n$, statement holds.

Suppose T has n vertices. Find a leaf v & delete.

- v had degree 1, so we delete 1 edge.
- $T-v$ is a tree, w/ $n-1$ vertices $\rightarrow$ $n-2$ edges.
- put v back: total edges $= n-2+1 = n-1$

□

Proved: if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$

Also true: for connected $G=(V,E)$ with $V \geq 1$

if $E = V-1$ then G is a tree

- What if $E < V-1$? $\rightarrow$ G isn't connected
- What if $E > V-1$? $\rightarrow$ G isn't acyclic