GRAPH CONNECTIVITY

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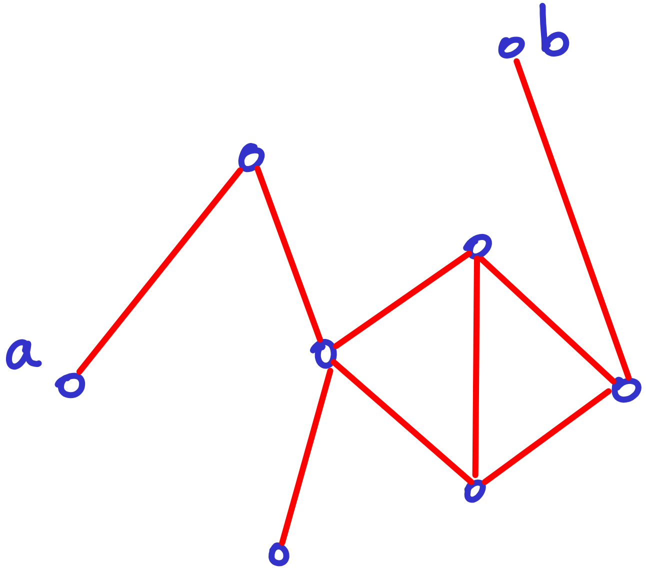

A **walk** is a sequence of vertices

$v_i, v_{i+1}, v_{i+2}, \ldots, v_k$

s.t. every v_j, v_{j+1} is an edge

$(i \le j < k)$

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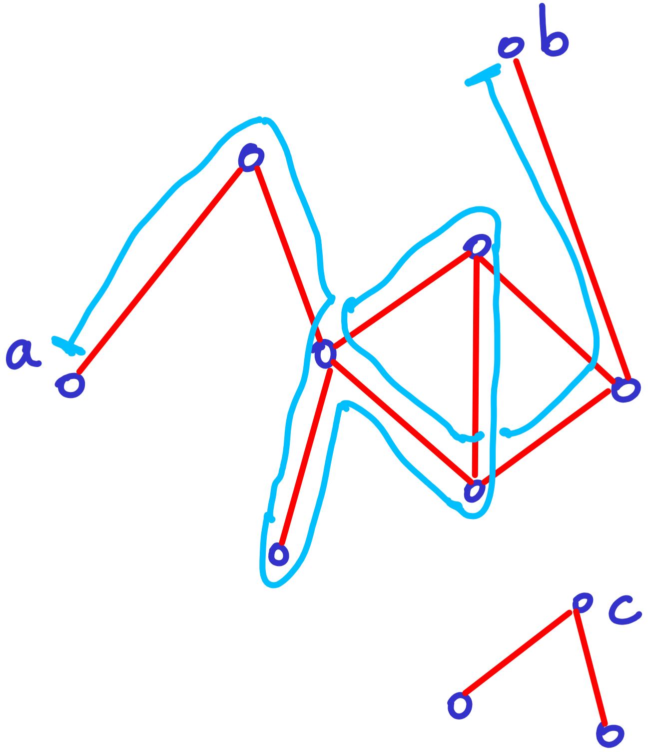

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We can walk from a to b, but not from a to c.

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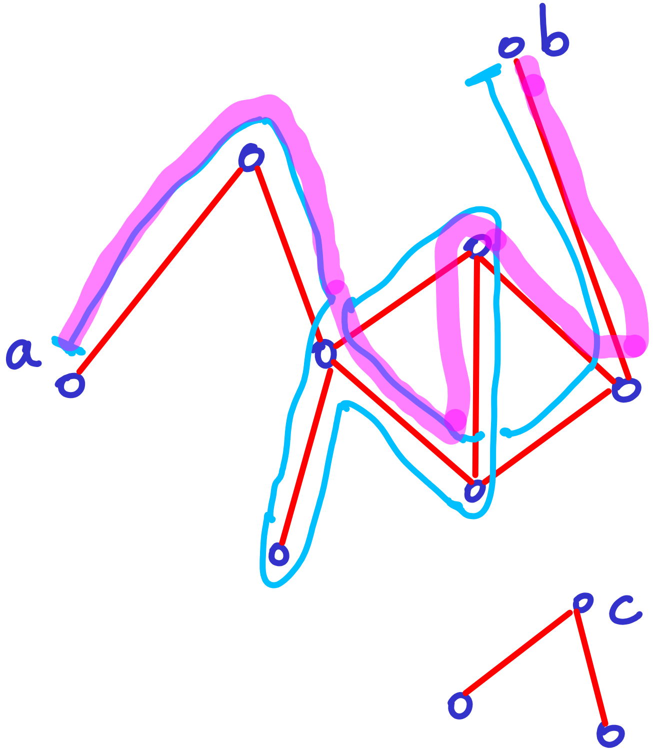

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s.t. every v_j, v_{j+1} is an edge $(i \le j < k)$

We can walk from a to b, but not from a to c.

A **path** is a walk with distinct vertices

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A vertex x is connected to a vertex y (in G)
if there is a path from x to y

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If there is an x-y walk in G then there is an x-y path in G.

[proof ?]

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A vertex x is connected to a vertex y (in G)
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Proof by contradiction: Assume $\exists$ x-y walk, $\nexists$ x-y path...

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If there is an x-y walk in G then there is an x-y path in G.

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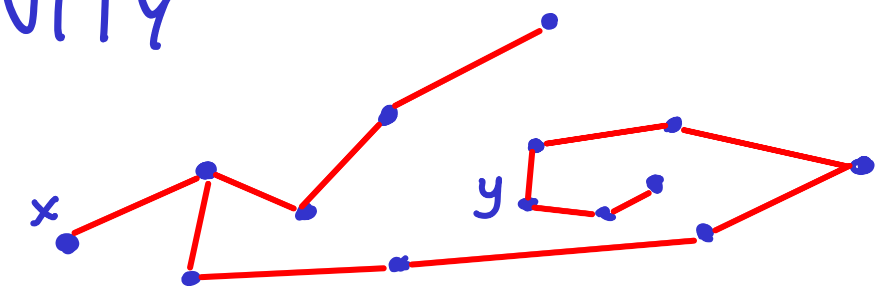
CONTRADICTION

$\square$

GRAPH CONNECTIVITY

A graph is connected

if every pair of vertices is connected



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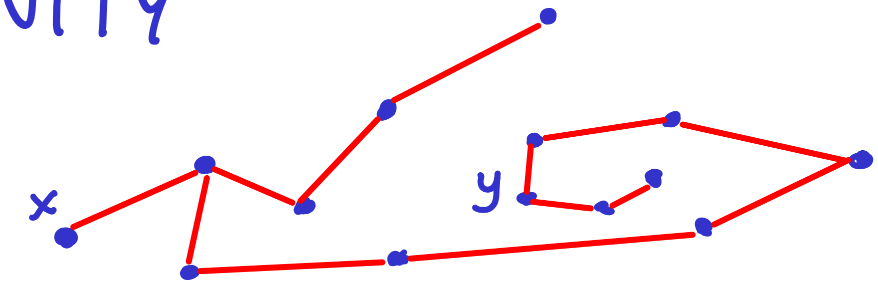

A graph is <u>connected</u>

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If vertex x is connected to vertex y

then they are in the same <u>component</u>.

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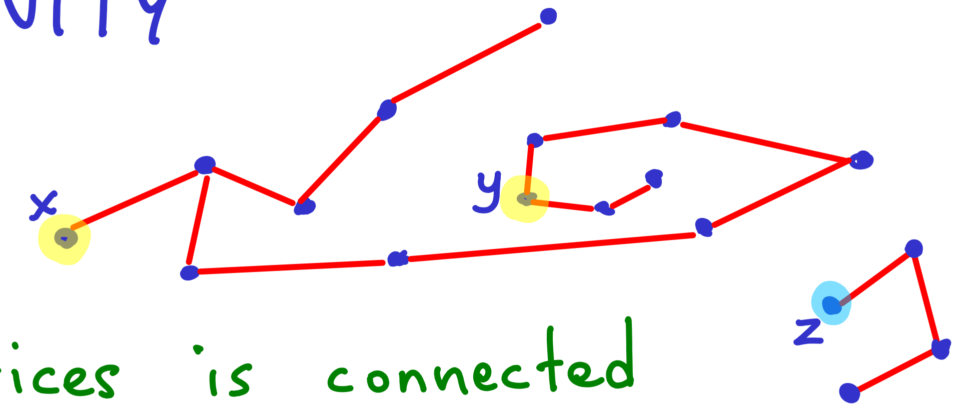

A graph is connected

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If vertex x is connected to vertex y

then they are in the same component.

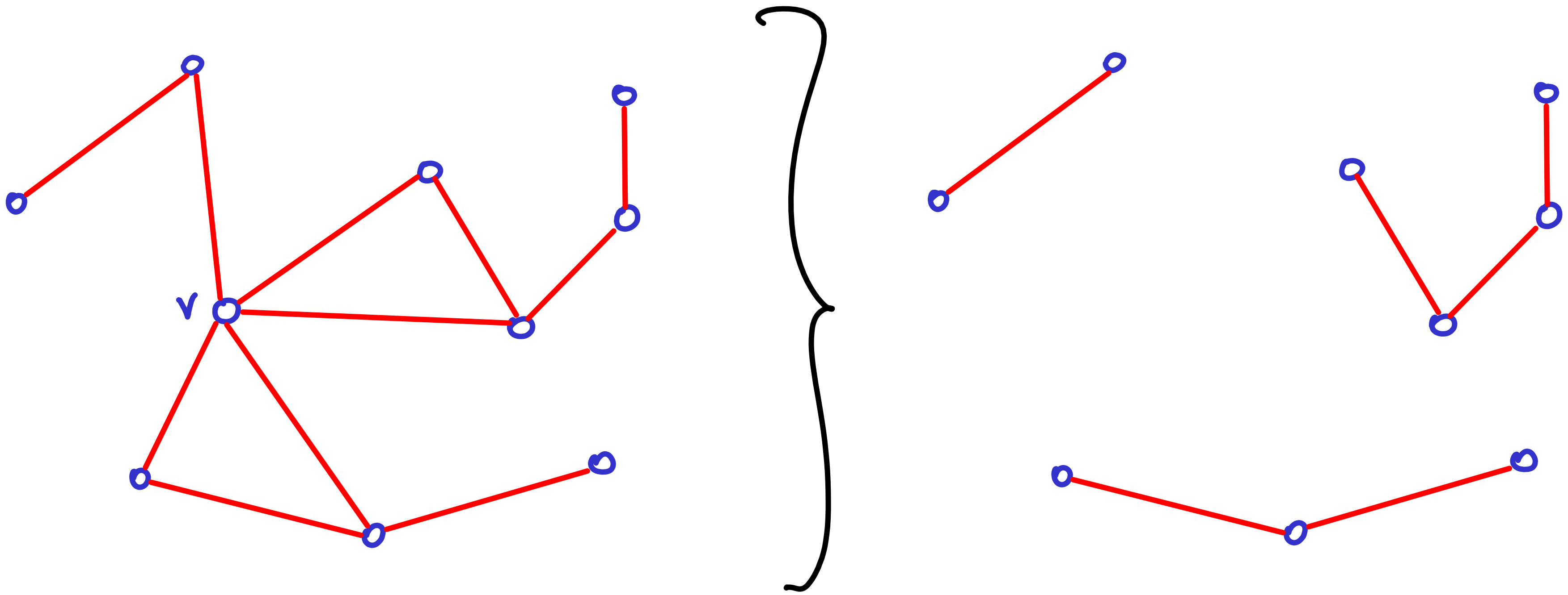
If x & y are in the same component

but x & z are not

then y & z are not.

Given G, remove a vertex: $G - v$

If $G - v$ has more components than G, then v is a cut vertex.



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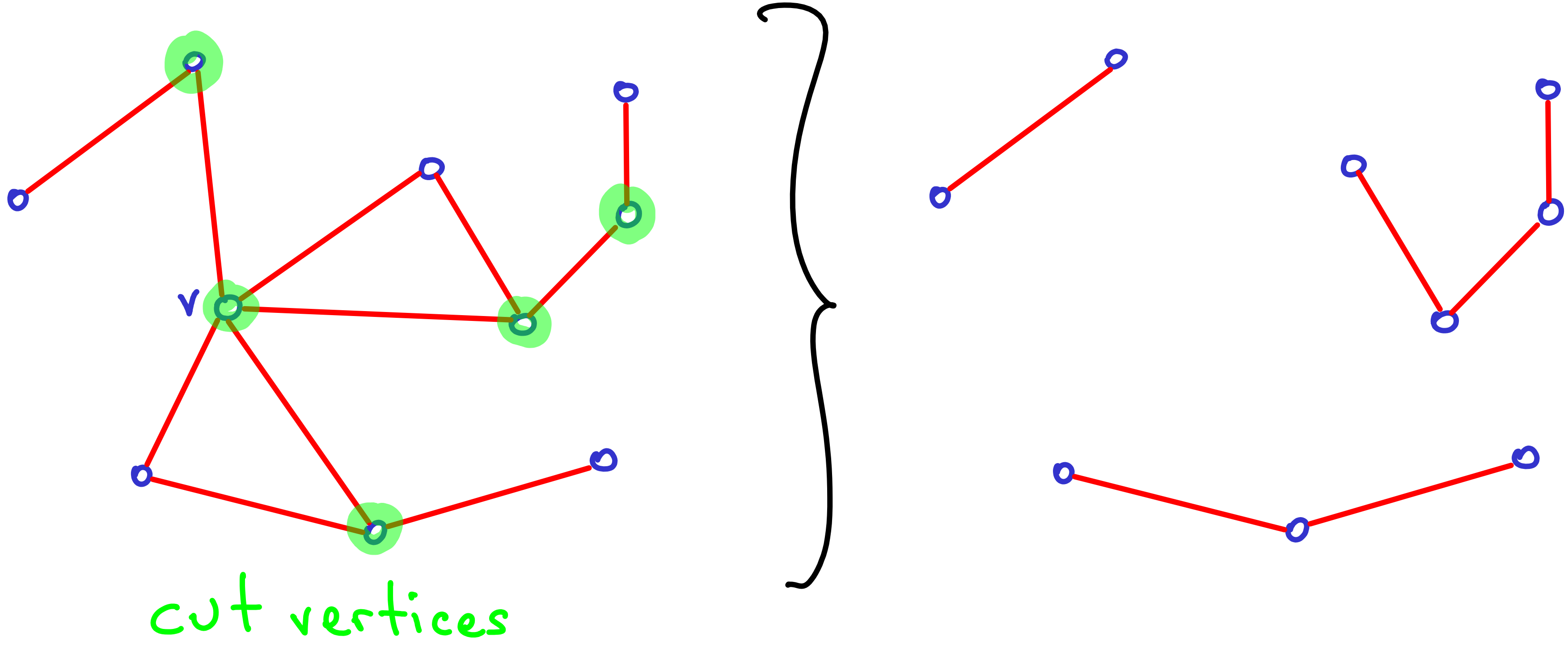
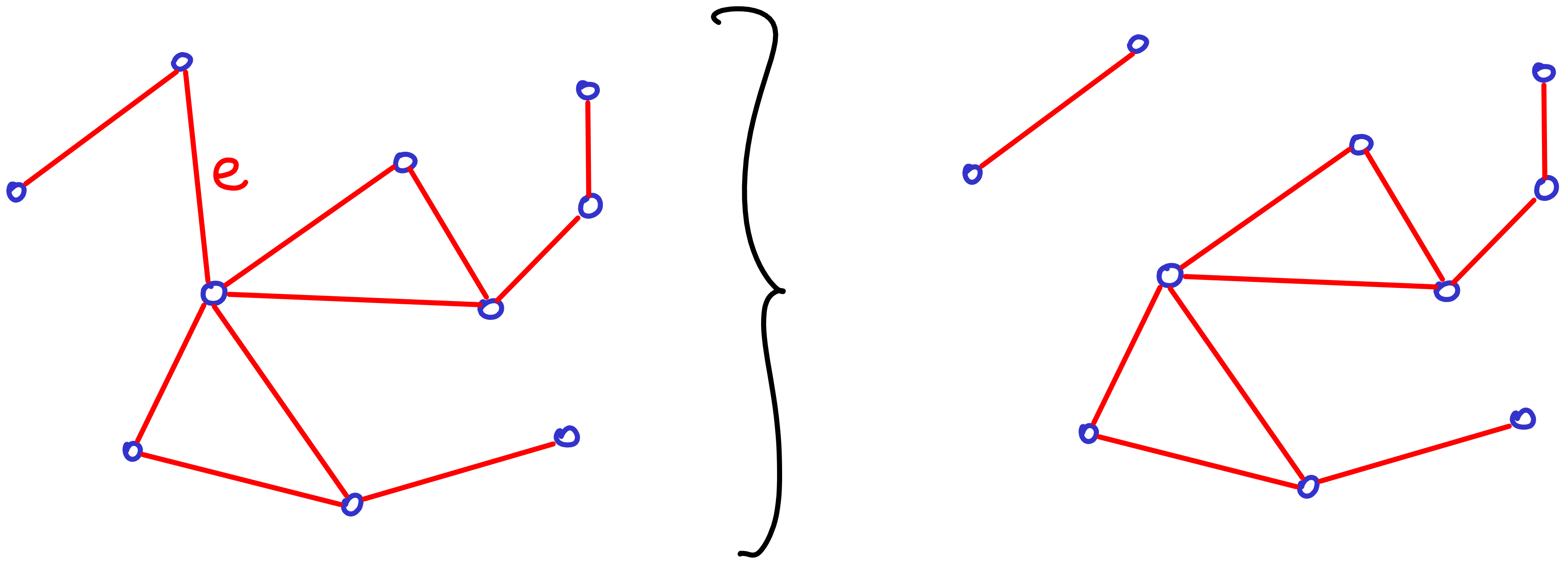


Given G, remove an edge : $G-e$

If $G-e$ has more components than G, then e is a cut edge.



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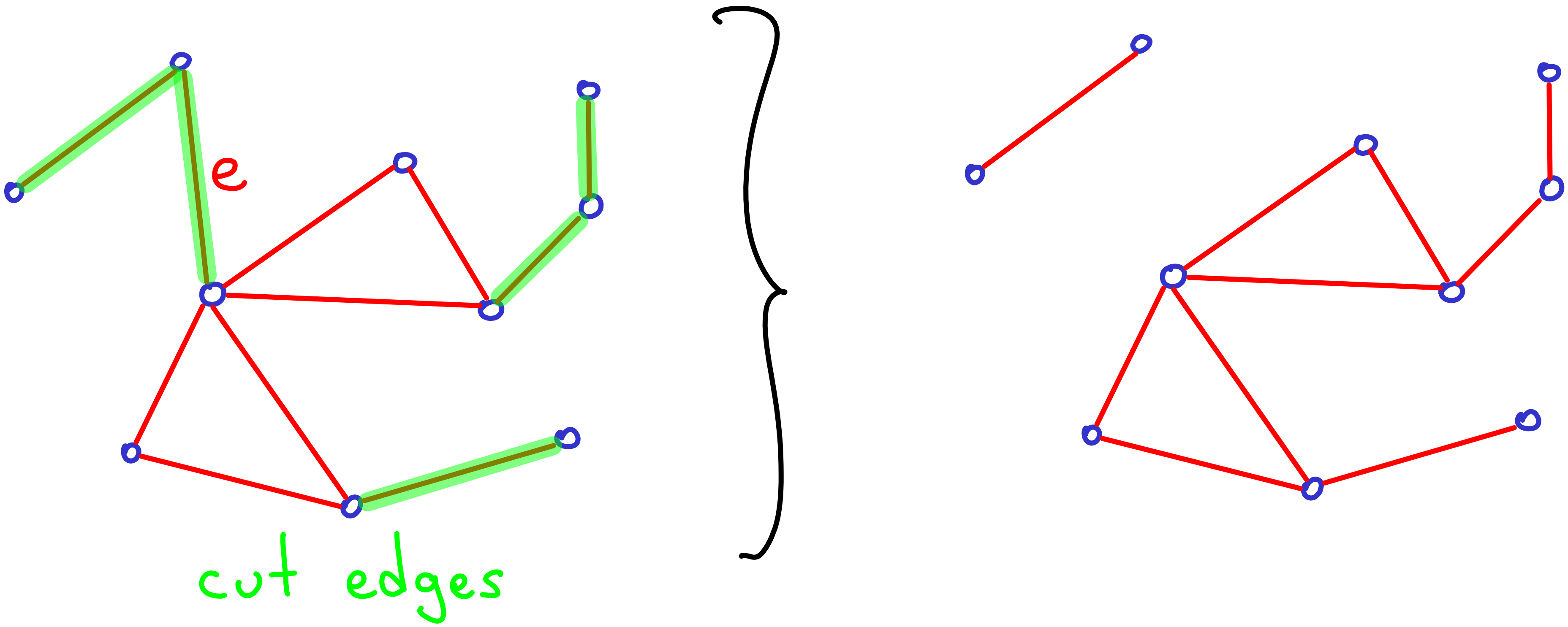

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(a cycle is a path w/ start = end)

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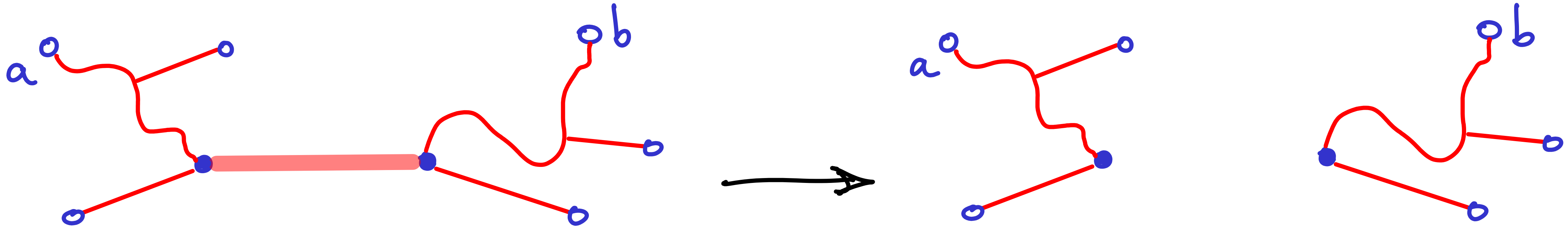

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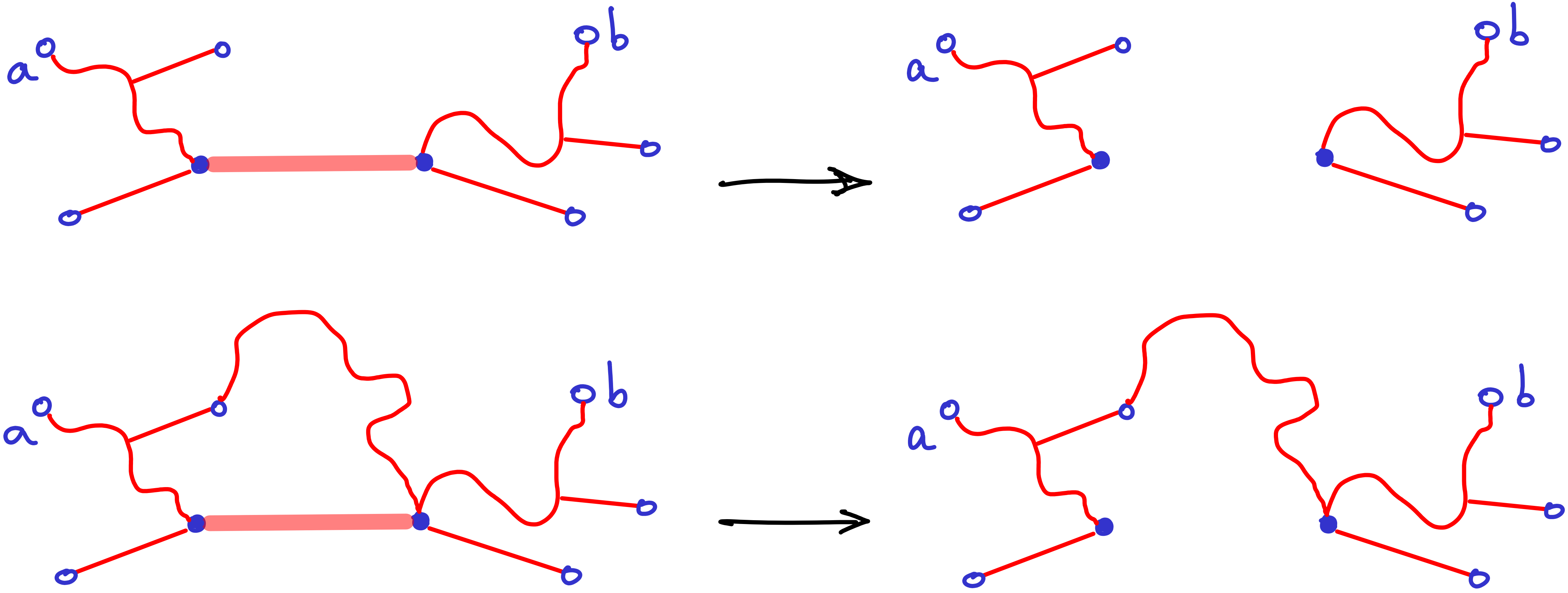
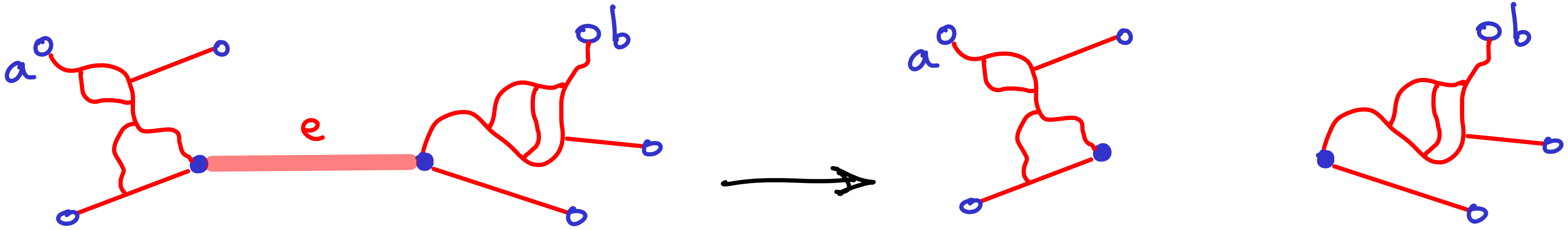


... cont'd : if e is a cut edge then $\exists$ a, b s.t.

a is not connected to b in $G - e$.

(but they are connected in G)



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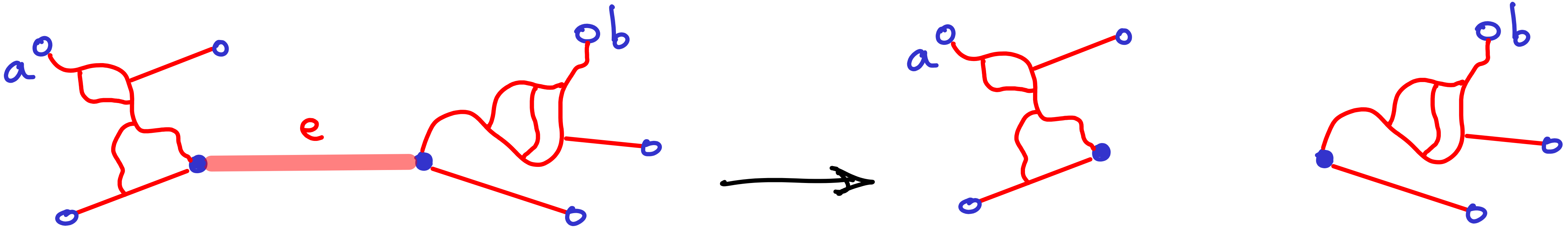
So all paths from a to b in G use e



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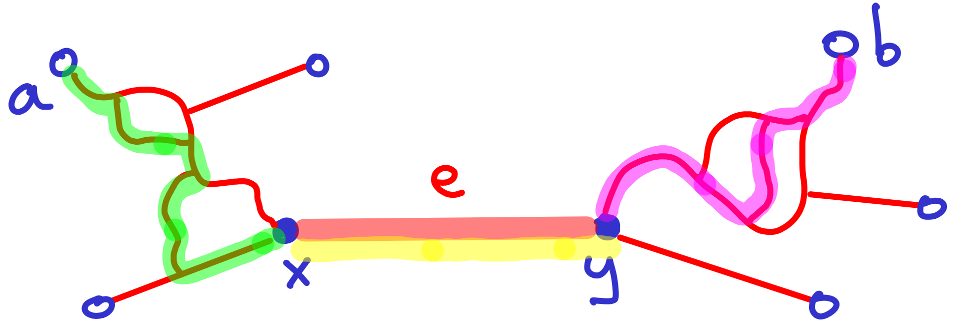
So all paths from a to b in G use e, i.e., $\exists$ path $a \cdots x y \cdots b$ where $e = \overline{xy}$.



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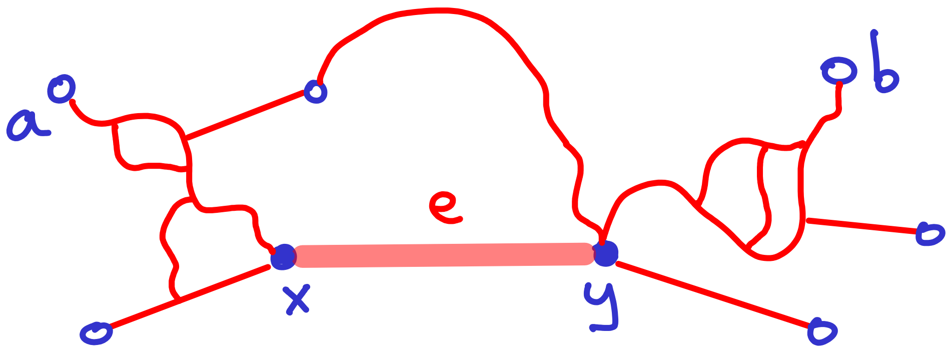

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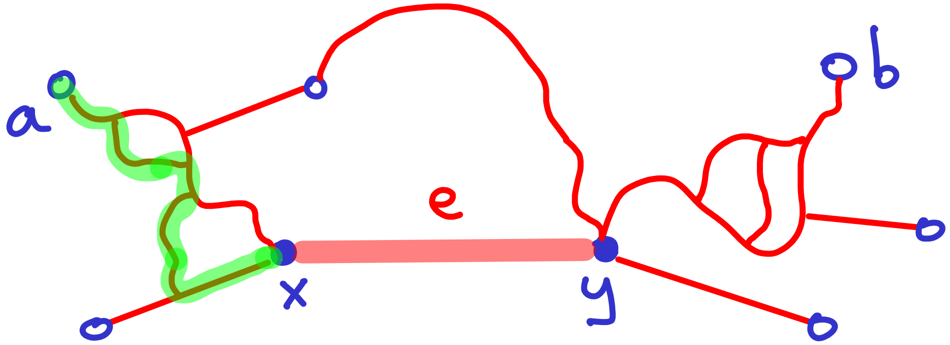

But if e is on a cycle

then we can walk from a to x,

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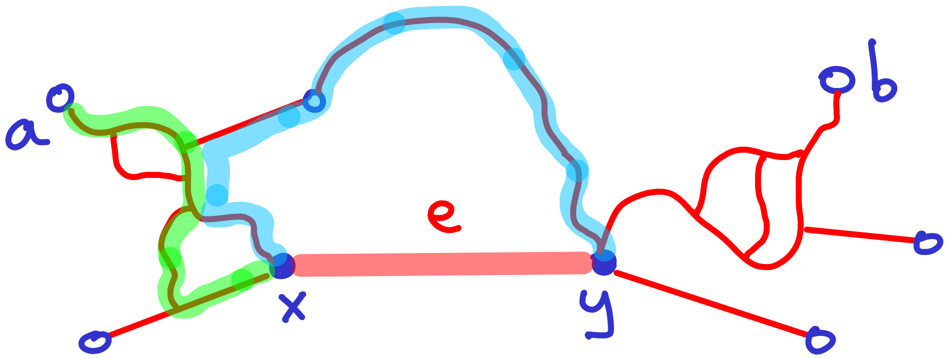

But if e is on a cycle then we can walk from a to x, walk from x to y, without using e,

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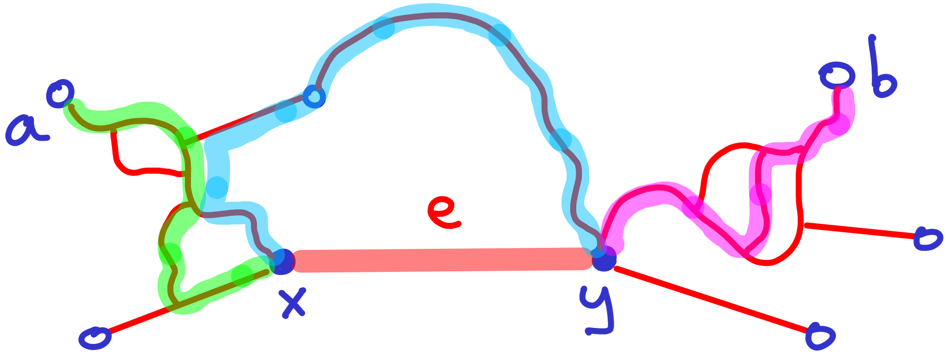

But if e is on a cycle then we can walk from a to x, walk from x to y, without using e, and walk from y to b.

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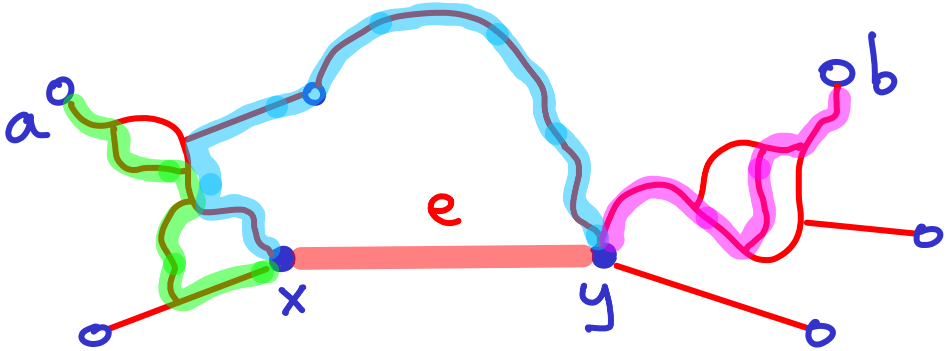

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CONTRADICTION

$\square$

observation

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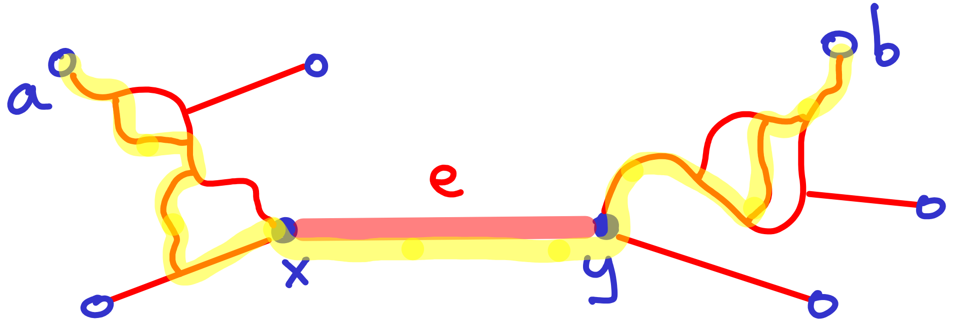

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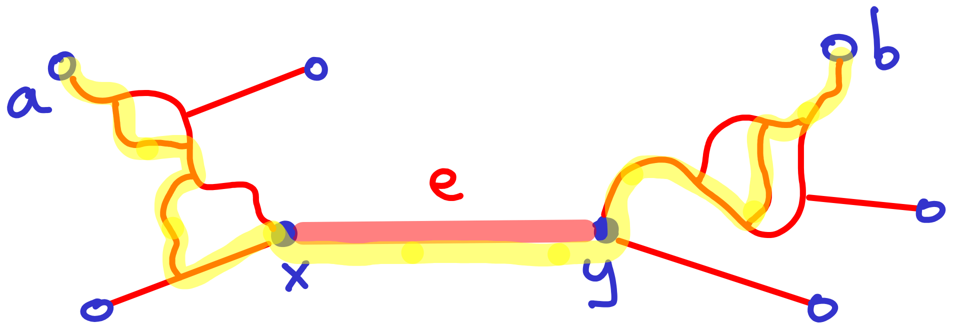

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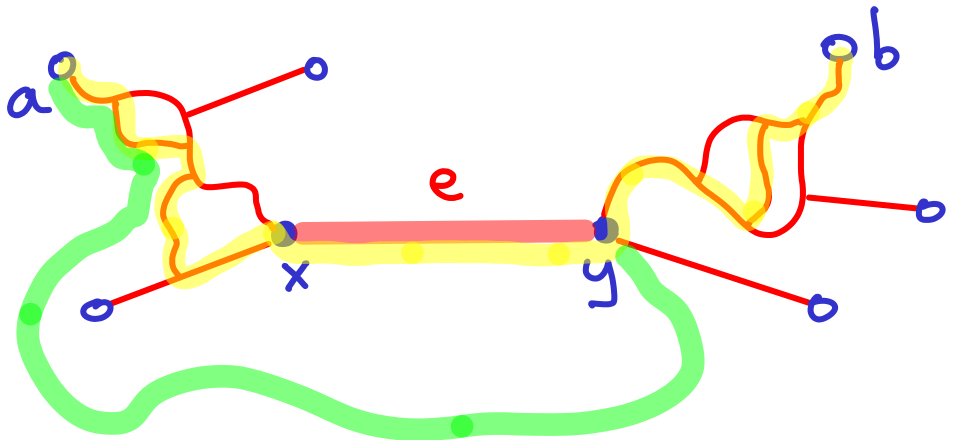
would create a cycle



$\square$

Claim: Removing a cut edge $e = (x, y)$ increases the number of components by 1.

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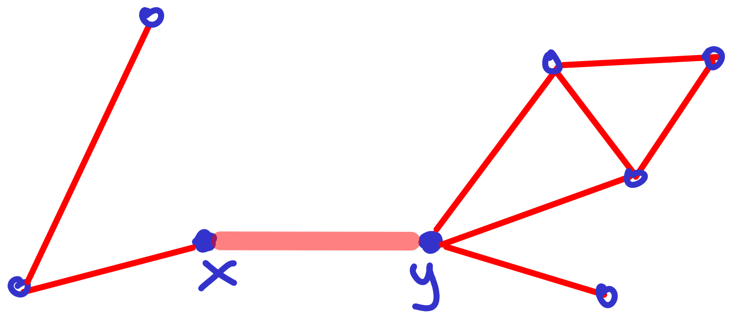
* e can only affect the component it's in. So focus on connected graphs.



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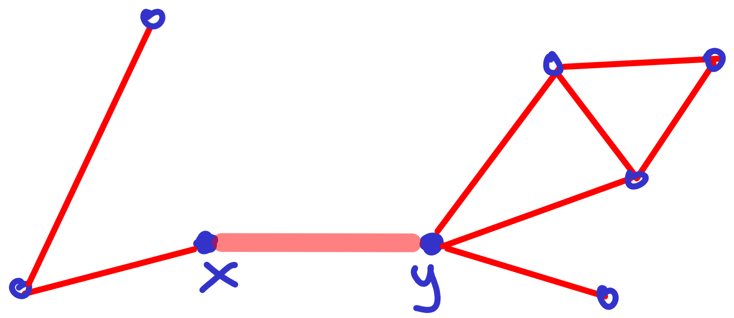

By definition of cut edge, we get at least 1 new component.

The question is: why not >1 ?

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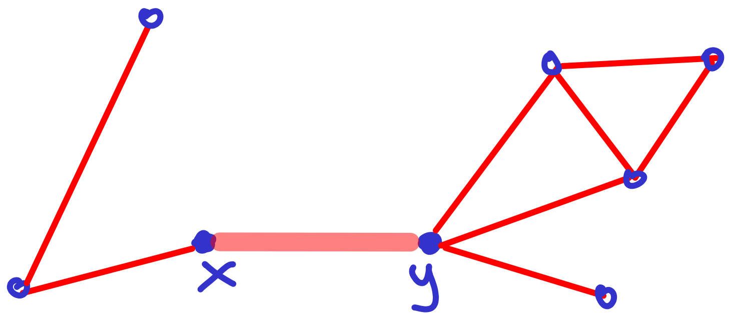

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Proof by contradiction.

Suppose $G - e$ has $\underbrace{\geq 3}_{\geq 2 \text{ new}}$ components.

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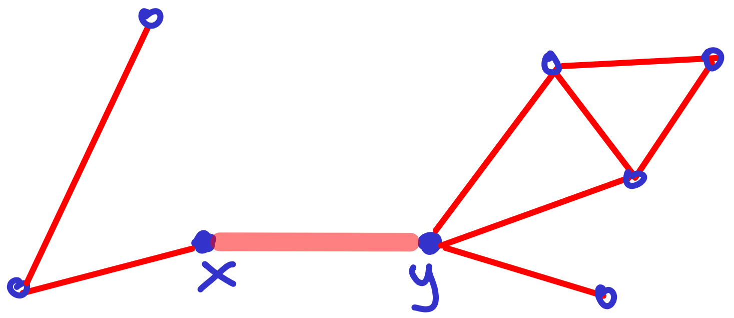

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Suppose $G - e$ has ≥ 3 components. $\exists\, a, b, c$ in different components.

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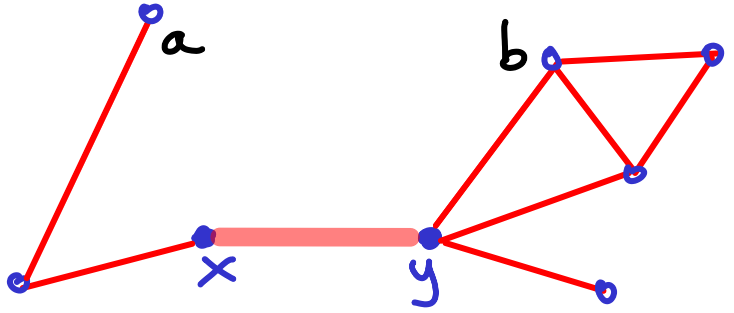

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Suppose $G - e$ has ≥ 3 components. $\exists\, a, b, c$ in different components.

In G, all paths $a \to b$ use e } wlog $a \to x \to y \to b$

Claim: Removing a cut edge $e=(x,y)$ increases the number of components by 1.

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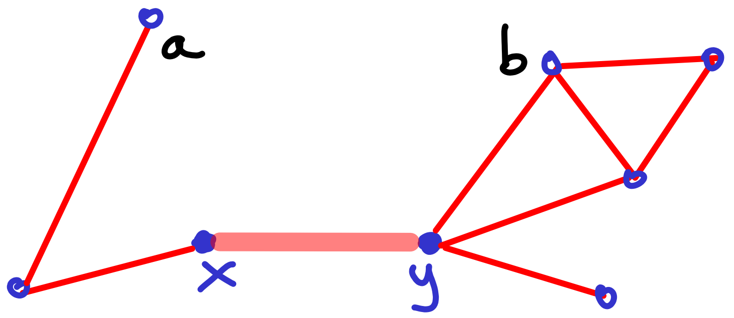

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Suppose $G-e$ has $\geqslant 3$ components. $\exists\, a,b,c$ in different components.

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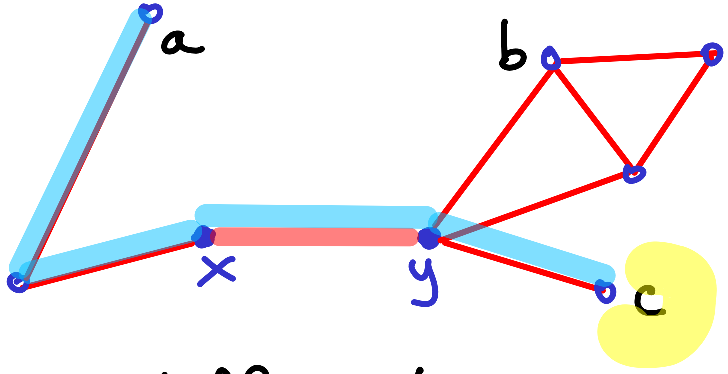

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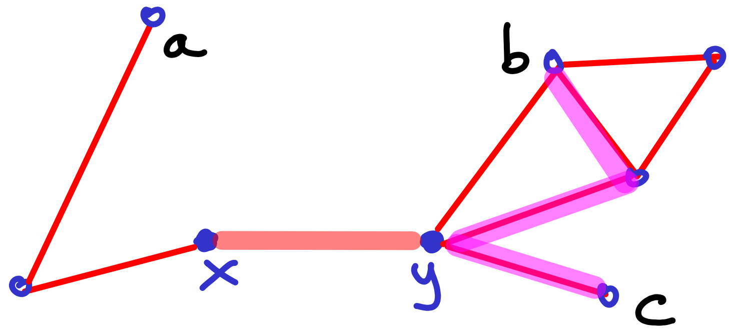

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$\Rightarrow$ b & c in same component

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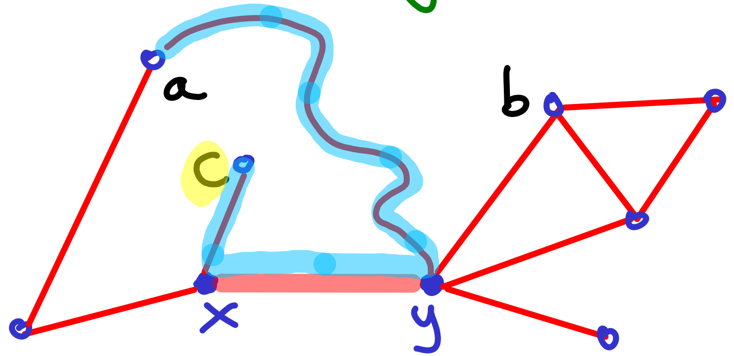

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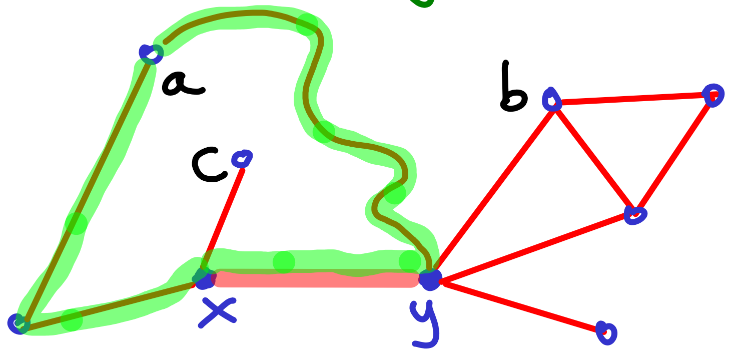

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Proof by contradiction.

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all paths $a \to c$ use e: if $a \to x \to y \to c$ ⇒ b & c in same component

if $a \to y \to x \to c$ ⇒ e: not a cut edge

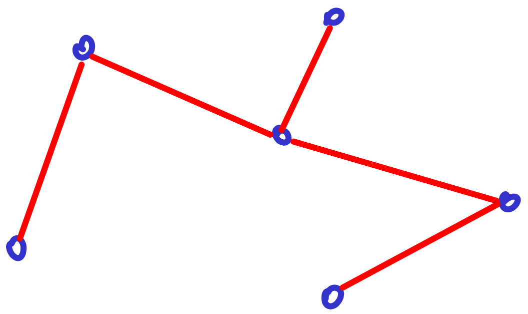
$\square$

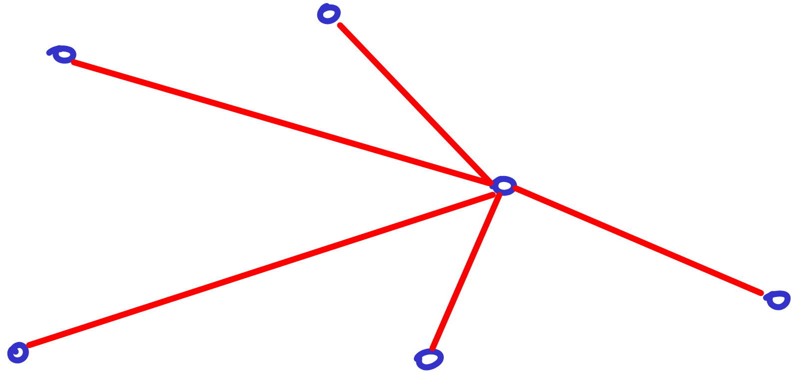
Recap

1) A cut edge can't be on a cycle.

2) Removing a cut edge $e = (x, y)$ increases the number of components by 1.

TREES : CONNECTED ACYCLIC GRAPHS





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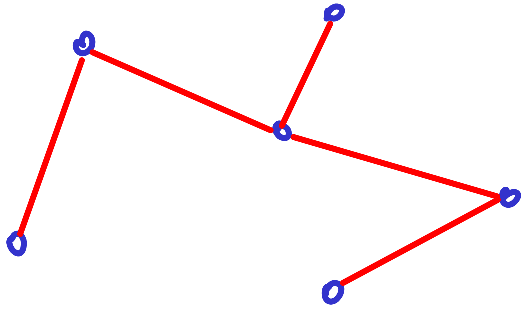
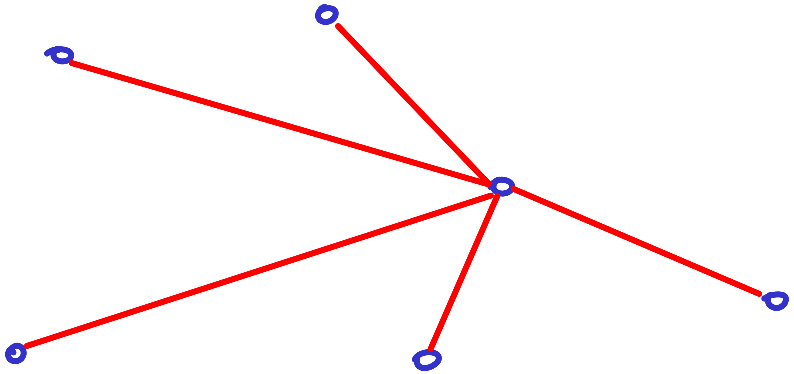


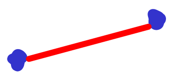


FORESTS : ACYCLIC GRAPHS (collections of trees)

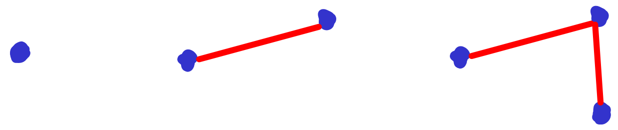
$V = 1$

•

$V = 1$ $V = 2$



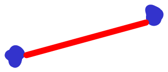
$V = 1$ $\quad$ $V = 2$ $\quad$ $V = 3$



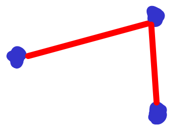
(3 isomorphs)

$V = 1$

$V = 2$

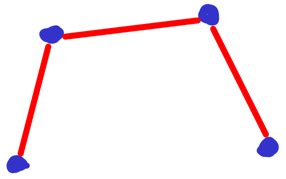


$V = 3$

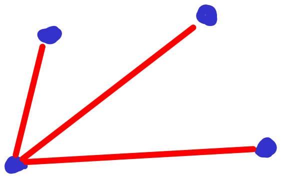


(3 isomorphs)

$V = 4$

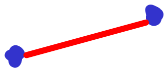


vs

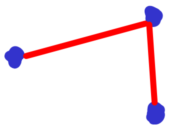


V = 1

V = 2

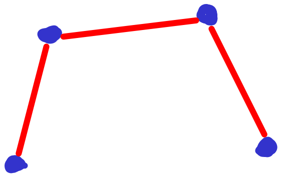


V = 3

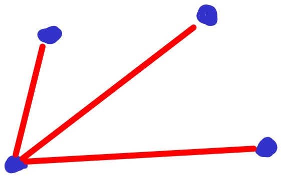


(3 isomorphs)

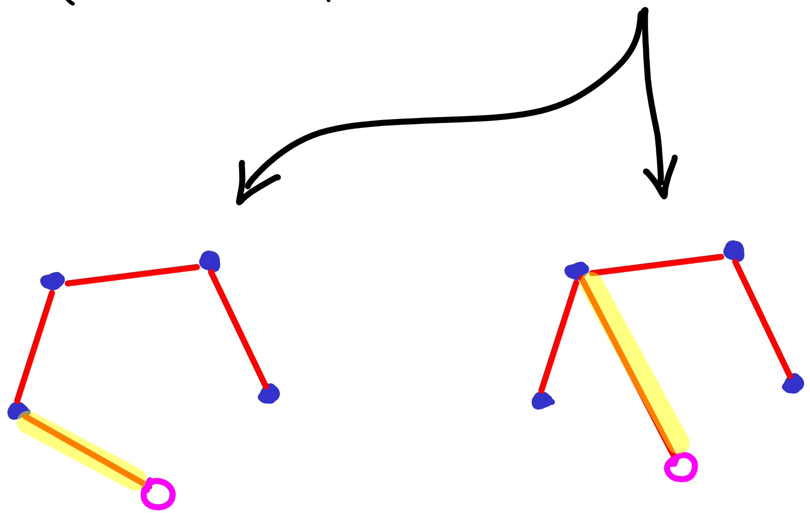
V = 4

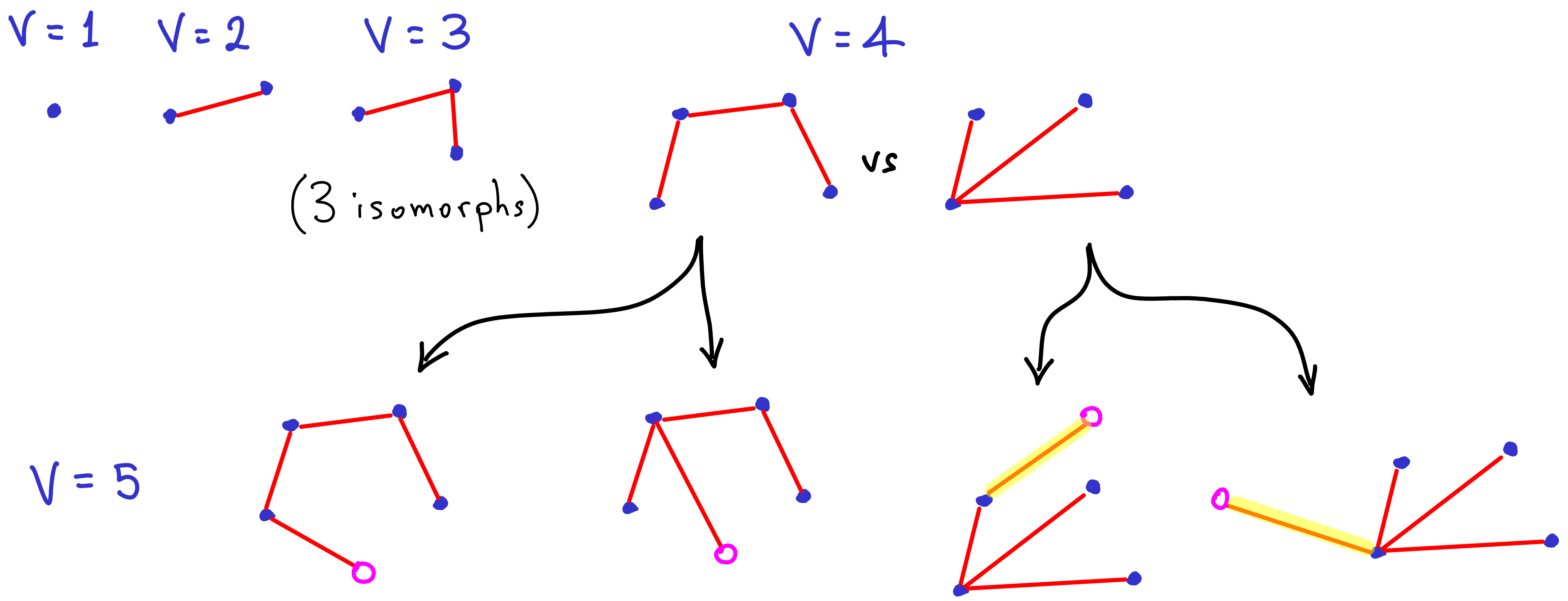


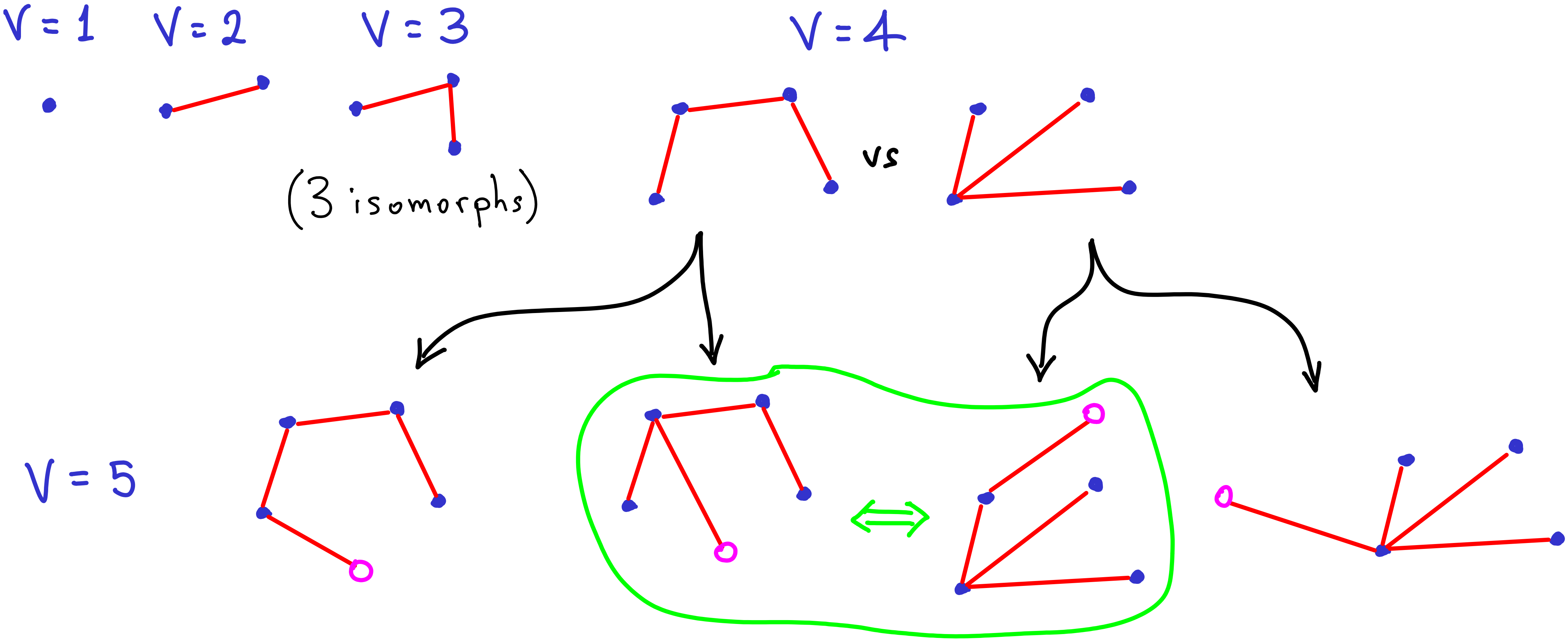
vs



V = 5




V = 1
V = 2
V = 3
(3 isomorphs)
V = 4
vs
V = 5


V = 1
V = 2
V = 3
(3 isomorphs)
V = 4
vs
V = 5

tree $\overset{\Longleftrightarrow}{\text{claim}}$ there is a unique path between every pair of vertices

tree $\iff$ there is a unique path between every pair of vertices

---

$\Rightarrow$

prove: IF tree THEN "$\exists$ unique path..."

tree $\Longleftrightarrow$ there is a unique path between every pair of vertices

---

$\Rightarrow$ • for any vertices a, b : a path exists (trees are connected)

next?

tree $\Longleftrightarrow$ there is a unique path between every pair of vertices

$\Rightarrow$

- for any vertices a,b : a path exists (trees are connected)
- suppose $\geqslant 2$ paths.

tree $\iff$ there is a unique path between every pair of vertices

$\Rightarrow$

- for any vertices a, b : a path exists (trees are connected)
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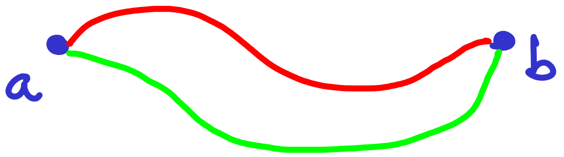

tree $\iff$ there is a unique path between every pair of vertices

$\Rightarrow$ • for any vertices a, b : a path exists (trees are connected)

• suppose ≥ 2 paths.

tree $\iff$ there is a unique path between every pair of vertices

$\Rightarrow$

- for any vertices a, b : a path exists (trees are connected)
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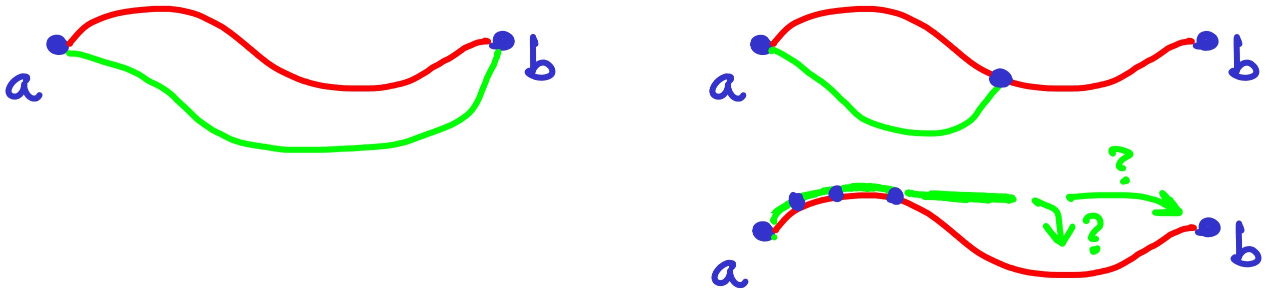

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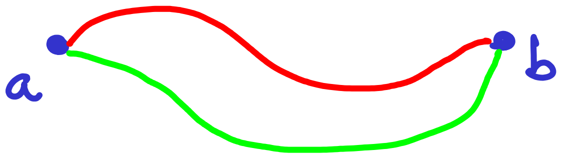

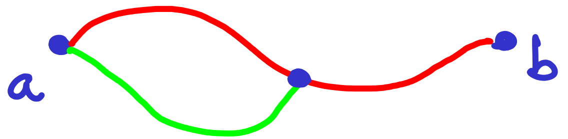

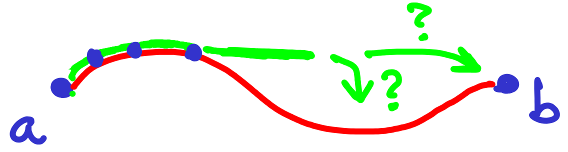

$\rightarrow$ cycle : contradiction of tree: acyclic

tree $\Longleftrightarrow$ there is a unique path between every pair of vertices

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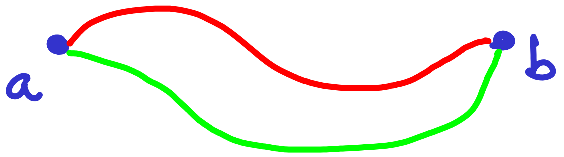
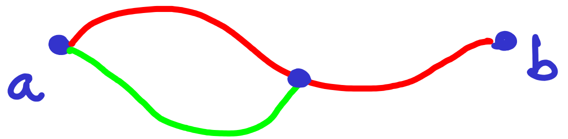
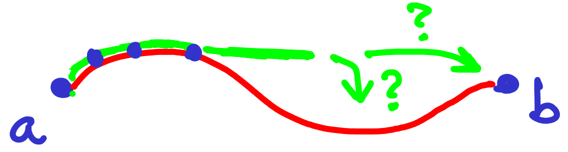






cycle : contradiction of tree: acyclic

$\Leftarrow$ What 2 properties do we need to prove?

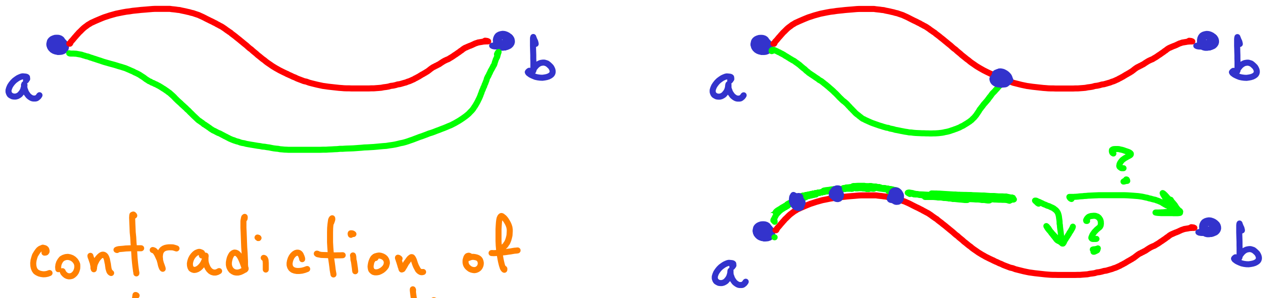
IF "$\exists$ unique path..." THEN tree

tree $\iff$ there is a unique path between every pair of vertices

---

$\Rightarrow$ • for any vertices a,b : a path exists (trees are connected)

• suppose ≥ 2 paths.



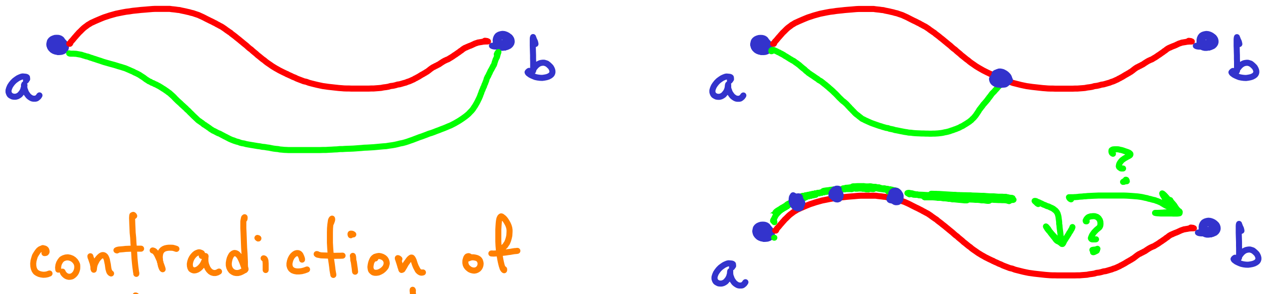
$\hookrightarrow$ cycle : contradiction of tree: acyclic

$\Leftarrow$ • if for every 2 vertices a path exists, then graph is connected

tree $\iff$ there is a unique path between every pair of vertices

$\Rightarrow$ • for any vertices a, b : a path exists (trees are connected)

• suppose ≥ 2 paths.



↳ cycle : contradiction of tree : acyclic

$\Leftarrow$ • if for every 2 vertices a path exists, then graph is connected

• if any 2 vertices are on a unique path, they are not on a cycle.
↳ acyclic

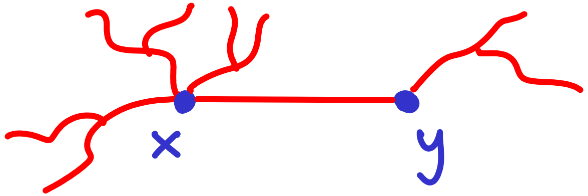
$\square$

For any connected graph,

tree $\overset{}{\underset{\text{claim}}{\Longleftrightarrow}}$ every edge is a cut edge

For any connected graph,

tree $\iff$ every edge is a cut edge

$\Rightarrow$ 

$\Big\}$ for any $\overline{xy}$ $\Big\}$ tree $\Rightarrow$ unique path from x to y

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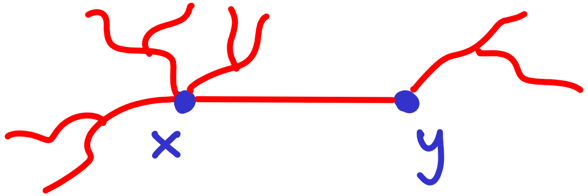
$\Rightarrow$ 
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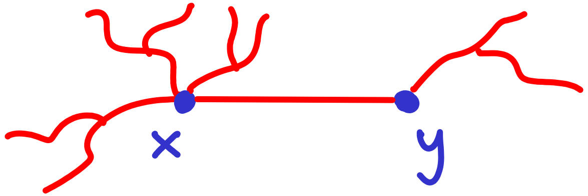
$\Rightarrow$ 

for any $\overline{xy}$

tree $\Rightarrow$ unique path from x to y

$\Rightarrow$ $\overline{xy}$: cut edge

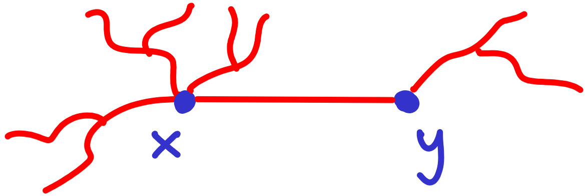
$\Leftarrow$ suppose graph $\neq$ tree. then...?

even though every edge is a cut edge

For any connected graph,

tree $\Longleftrightarrow$ every edge is a cut edge

$\Rightarrow$



$\Bigg\}$ for any $\overline{xy}$ $\Bigg\}$ tree $\Rightarrow$ unique path from x to y

$\Rightarrow$ $\overline{xy}$: cut edge

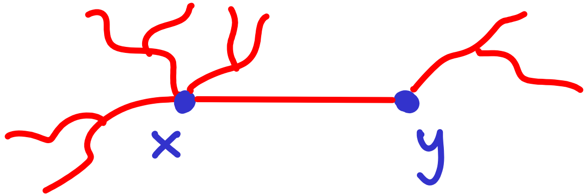
$\Leftarrow$ suppose graph $\neq$ tree.

Then it has a cycle.

...?

For any connected graph,

tree $\iff$ every edge is a cut edge

$\Rightarrow$ 

for any $\overline{xy}$

tree $\Rightarrow$ unique path from x to y

$\Rightarrow$ $\overline{xy}$: cut edge

$\Leftarrow$ suppose graph $\neq$ tree.

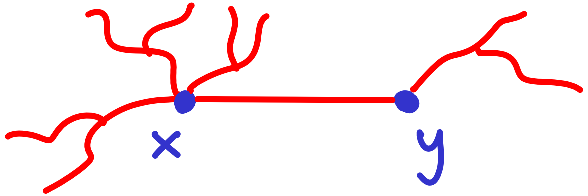
Then it has a cycle.

We have proved: A cut edge can't be on a cycle.

so?

For any connected graph,

$$\text{tree} \iff \text{every edge is a cut edge}$$

$\Rightarrow$ 

for any $\overline{xy}$: tree $\Rightarrow$ unique path from x to y

$\Rightarrow$ $\overline{xy}$: cut edge

$\Leftarrow$ suppose graph $\neq$ tree.

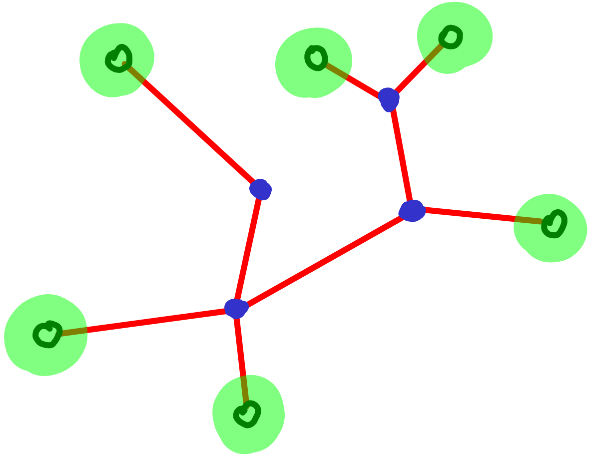
Then it has a cycle.

We have proved: A cut edge can't be on a cycle.

not every edge is a cut edge. (CONTRADICTION)

$\square$

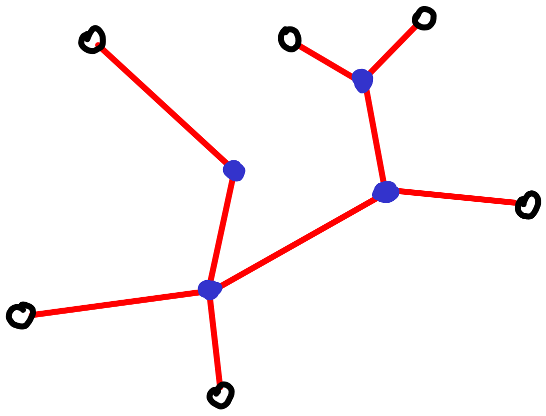
LEAVES : vertices of degree 1



LEAVES : vertices of degree 1

Claim:

If $V \geqslant 2$, then T has $\geqslant 2$ leaves

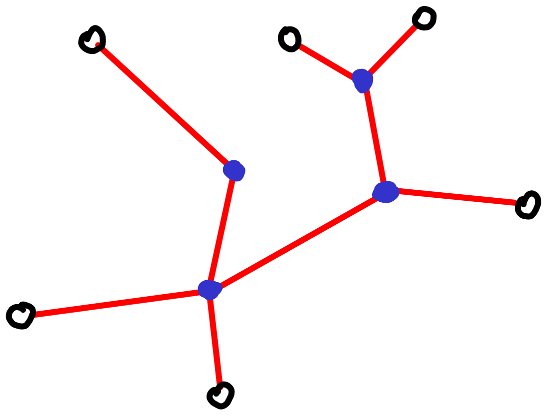


LEAVES : vertices of degree 1

If $V \geqslant 2$, then T has $\geqslant 2$ leaves

Consider longest path in T. $v_1 \ldots v_k$

$(k \geqslant 2)$



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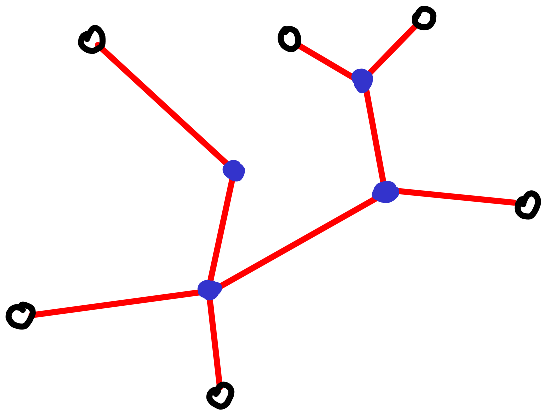
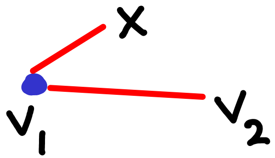


Consider longest path in T. $v_1 \ldots v_k$

$(k \geqslant 2)$

If $v_1 \neq$ leaf, then



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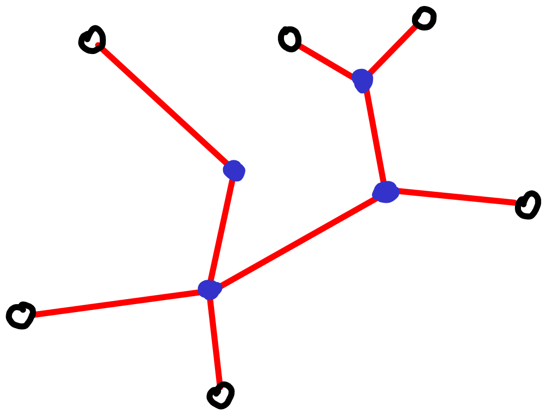

Consider longest path in T. $v_1 \dots v_k$

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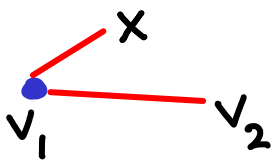
If $v_1 \neq$ leaf, then

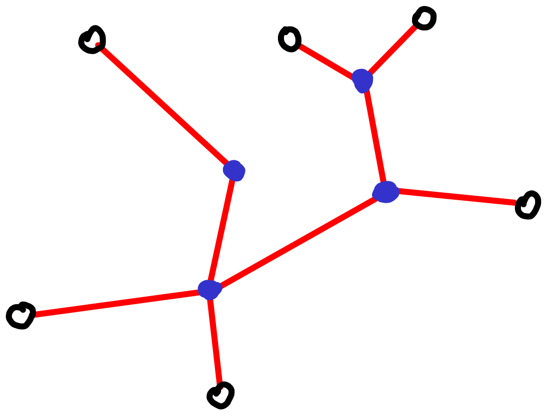


$x \neq v_i$ (not on path)

why?

LEAVES : vertices of degree 1

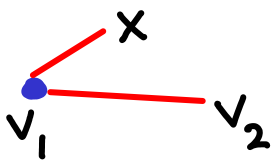
If $V \geq 2$, then T has ≥ 2 leaves



Consider longest path in T. $v_1 \dots v_k$

$(k \geq 2)$

If $v_1 \neq$ leaf, then

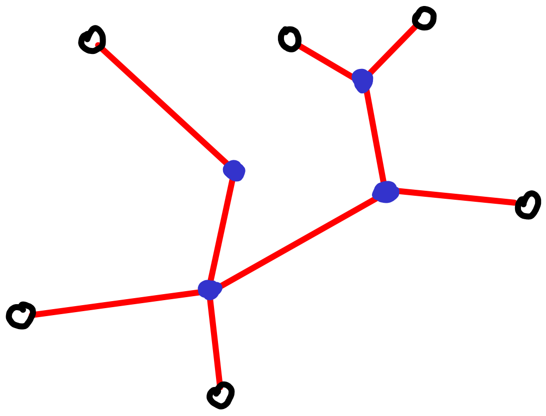


$x \neq v_i$ (not on path)

($x = v_i$ would create cycle)

LEAVES : vertices of degree 1

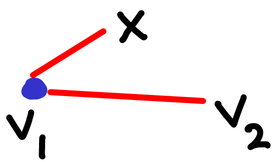
If $V \geq 2$, then T has ≥ 2 leaves



Consider longest path in T. $v_1 \dots v_k$

($k \geq 2$)

If $v_1 \neq$ leaf, then



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Then $x v_1 \dots v_k$: longer path

LEAVES : vertices of degree 1

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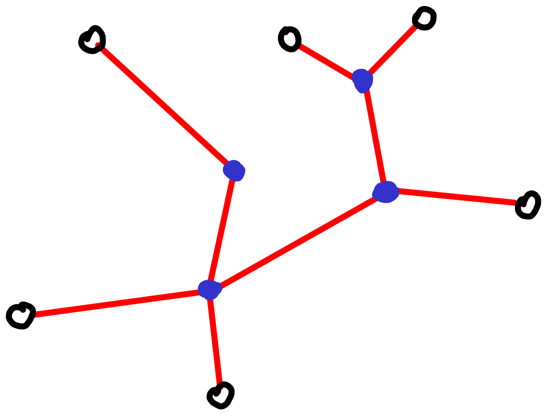

Consider longest path in T. $v_1 \dots v_k$

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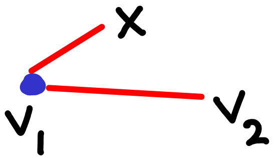
If $v_1 \neq$ leaf, then



$x \neq v_i$ (not on path)

($x = v_i$ would create cycle)

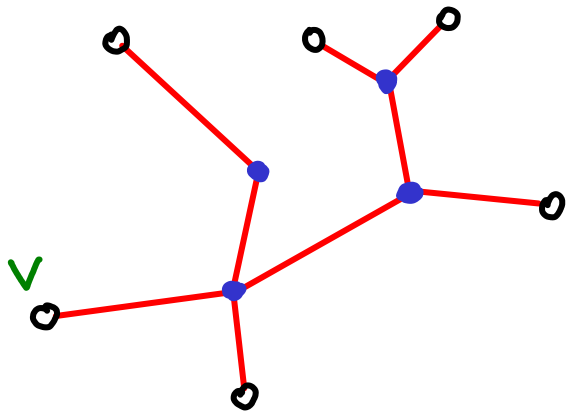
Then $x v_1 \dots v_k$: longer path : CONTRADICTION

So v_1 & v_k : leaves $\square$

claim: If v is a leaf in tree T, then $T-v$ is a tree

If v is a leaf in tree T, then T-v is a tree

- removing v doesn't create cycles.

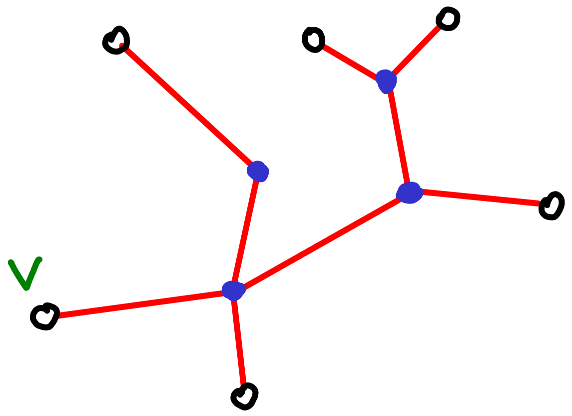


If v is a leaf in tree T, then T-v is a tree

- removing v doesn't create cycles.

- removing v doesn't disconnect.
  ($v \neq$ cut vertex $\rightarrow T-v$ is connected)

  why?



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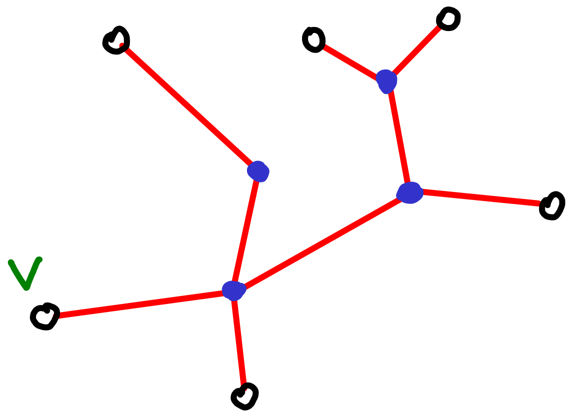

↳ if v were a cut vertex, then $\exists\, a,b$ $(a \neq v, b \neq v)$ s.t.
any path $a \rightarrow b$ must use v.

If v is a leaf in tree T, then $T-v$ is a tree

- removing v doesn't create cycles.
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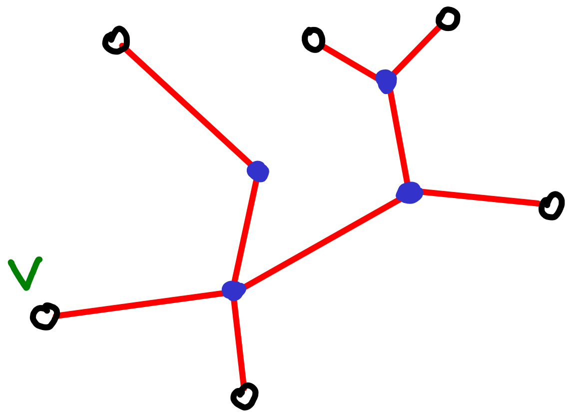
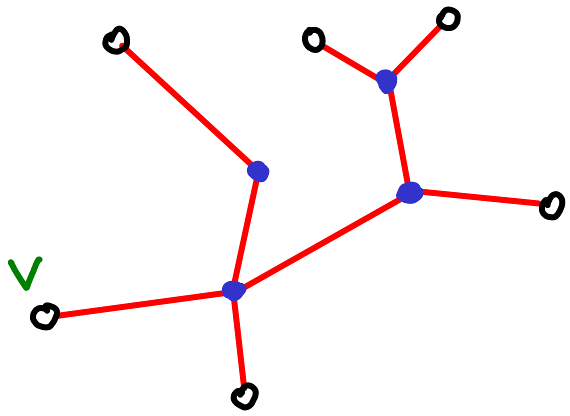


↳ if v were a cut vertex, then $\exists\, a,b$ $(a \neq v,\ b \neq v)$ s.t. any path $a \rightarrow b$ must use v.

in fact,
"the only path"
(T : unique paths)

If v is a leaf in tree T, then $T-v$ is a tree

- removing v doesn't create cycles.



- removing v doesn't disconnect.
  ($v \neq$ cut vertex $\rightarrow T-v$ is connected)

↳ if v were a cut vertex, then $\exists\, a,b$ $(a \neq v,\ b \neq v)$ s.t.
any path $a \rightarrow b$ must use v.
But v is a dead end:
can't be part of such a path

in fact, "the only path" (T: unique paths)

□

If v is a leaf in tree T, then $T-v$ is a tree

This allows us to use induction for:

if $|V(T)| = n \geqslant 2$ then $|E(T)| = n-1$

If v is a leaf in tree T, then $T-v$ is a tree

This allows us to use induction for:

if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$

pf: Base case: $n=2$  trivial

If v is a leaf in tree T, then T-v is a tree

This allows us to use induction for:

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Hypothesis: for $2 \leq k < n$, statement holds.

If v is a leaf in tree T, then $T-v$ is a tree

This allows us to use induction for:

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pf: Base case: $n=2$  trivial

Hypothesis: for $2 \leq k < n$, statement holds.

Suppose T has n vertices. Find a leaf v & delete.

If v is a leaf in tree T, then $T-v$ is a tree

This allows us to use induction for:

if $|V(T)| = n \geqslant 2$ then $|E(T)| = n-1$

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Suppose T has n vertices. Find a leaf v & delete.

- v had degree 1, so we delete 1 edge.

If v is a leaf in tree T, then T-v is a tree

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Suppose T has n vertices. Find a leaf v & delete.

- v had degree 1, so we delete 1 edge.
- T-v is a tree, w/ n-1 vertices $\rightarrow$ n-2 edges.

If v is a leaf in tree T, then $T-v$ is a tree

This allows us to use induction for:

if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$

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Hypothesis: for $2 \leq k < n$, statement holds.

Suppose T has n vertices. Find a leaf v & delete.

- v had degree 1, so we delete 1 edge.
- $T-v$ is a tree, w/ $n-1$ vertices $\rightarrow$ $n-2$ edges.
- put v back: total edges $= n-2+1 = n-1$

$\square$

Proved: if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$

Also true: for connected $G = (V, E)$ with $V \geq 1$

if $E = V-1$ then G is a tree

Proved: if $|V(T)| = n \geqslant 2$ then $|E(T)| = n-1$

Also true: for connected $G = (V, E)$ with $V \geqslant 1$

if $E = V-1$ then G is a tree

- What if $E < V-1$?

Proved: if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$

Also true: for connected $G = (V, E)$ with $V \geq 1$

if $E = V-1$ then G is a tree

- What if $E < V-1$? $\rightarrow$ G isn't connected

Proved: if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$

Also true: for connected $G = (V, E)$ with $V \geq 1$

if $E = V-1$ then G is a tree

- What if $E < V-1$? $\rightarrow$ G isn't connected

- What if $E > V-1$?

Proved: if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$

Also true: for connected $G = (V, E)$ with $V \geq 1$

if $E = V-1$ then G is a tree

- What if $E < V-1$? → G isn't connected
- What if $E > V-1$? → G isn't acyclic