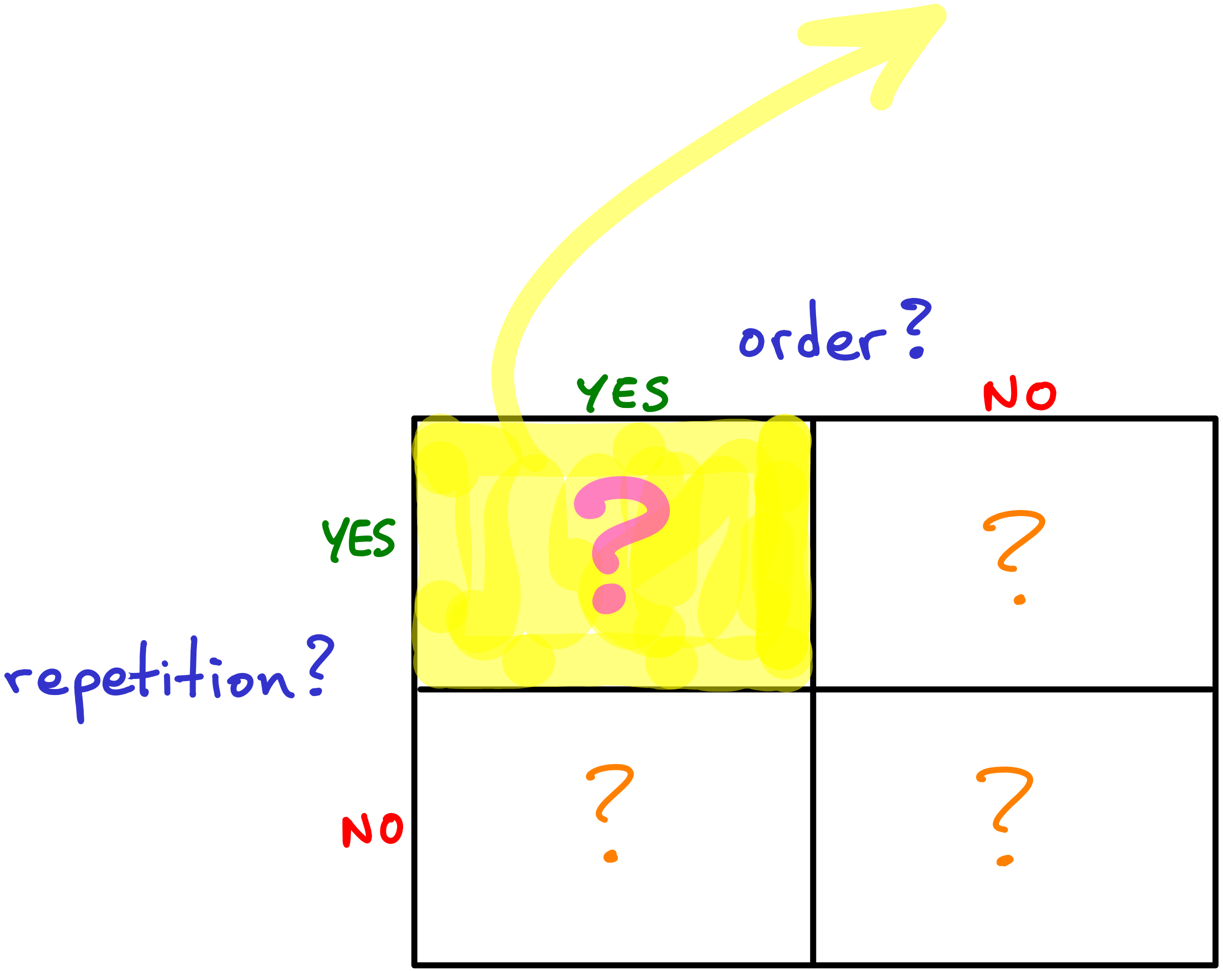
COMBINATIONS & PERMUTATIONS etc

n items to choose from, k choices to make

		does order matter?	
		YES	NO
is repetition allowed?	YES	?	?
	NO	?	?

n items, k choices : repetition allowed, order matters



order?

repetition?

	YES	NO
YES	?	?
NO	?	?

a simple multiplication rule

h hats, t shirts, s shoes, j jeans

colors

- how many ways to dress?

$h \cdot t \cdot s \cdot j$

choice 1 choice 4

- what if only 2 hats? → $n = 2$ items to choose from

↳ on 7 consecutive days? $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

$k = 7$ choices to make: choice 1 choice 7

↳ same as 7 people, 2 hats each

repetition allowed, order matters.


≠


$$2^7 = n^k$$

repetition? \ order?	YES	NO
YES	n^k	?
NO	?	?

n items to choose from, k choices to make

↳ repetition allowed, order matters

		order?	
		YES	NO
repetition?	YES	n^k	?
	NO	?	?

- how many binary numbers have length k?

  e.g., 100110, 000101, $\underbrace{000000}_{k=6}$

- how many binary numbers have length $\leq k$?

  e.g., $\underbrace{100110}_{k=6}$, 101, $\underset{<k}{0}$

n items to choose from, k choices to make

↳ repetition allowed, order matters

		order?	
		YES	NO
repetition?	YES	n^k	?
	NO	?	?

- how many binary numbers have length k? → 2^k

e.g., 100110, 000101, 000000

k ordered positions, for each we can choose $\overset{n=2}{\underbrace{0 \text{ or } 1}_{\text{can repeat choices}}}$

↳ like hats & days

- how many binary numbers have length ≤ k?

↳ context: no padding with 0's. e.g., $\underbrace{1001}_{k=6}$, 101, $\underset{<k}{0}$

consider lengths k, k-1, k-2, etc → disjoint counts, can add results

↳ for each scenario, start with 1.

but don't forget 0

	length		numbers	2^k 16
e.g.,	k=4:	1???	2^3	8
	3:	1??	2^2	4
	2:	1?	2^1	2
combine with nothing →	1:	1{ }	2^0	1
no 1's, combine with nothing →	0:	{ }{ }	1	1

n items to choose from, k choices to make

↳ repetition allowed, order matters

		order?	
		YES	NO
repetition?	YES	n^k	?
	NO	?	?

- how many decimal numbers have length k? → 10^k

$n = 10$

k ordered positions, for each we can choose $0, 1, 2, \ldots, 9$

- how many decimal numbers have length 0? → $10^k = 10^0 = 1$?

↳ choosing from n hats on 0 consecutive days. (absurd? 0? 1?)

Is it just a phrasing issue? A matter of definition?

Given the set of all decimal strings, how many elements have length 0?

↳ 1 empty string

How many ways can we form an ordered list of 0 items?

↳ 1 empty list

Know what choices are valid

- if numbers can't have length 0, require $k \geq 1$
- for strings, lists, sets, $k=0$ is ok

n items to choose from, k choices to make

↳ repetition allowed, order matters

		order?	
		YES	NO
repetition?	YES	n^k	?
	NO	?	?

- how many ways to assign k tasks to a group of n people?
  - each task is handled by one person.
    ↳ first decide who does task 1, then decide for task 2, etc
  - one person could handle multiple tasks.
    ↳ the task group (committee) has $\leq k$ people

doing nothing counts as something

- how many ways to assign 0 tasks to a group of $n \geq 1$ people? → $n^0 = 1$
- how many ways to assign 0 tasks to a group of 0 people? → $0^0 = 1$

combinatorics definition

n items to choose from, k choices to make

does order matter?

is repetition allowed?

	YES	NO
YES	n^k	?
NO	?	?

n items, k choices: repetition not allowed, order matters. $(k \leq n)$

easy: $k=1$: n ways to choose 1 item from n

familiar: $k=n$: $n!$ ways to choose n ordered items from n

- 7 hats, 7 days, must wear a different one each day → $7!$

  7 options on day 1, 6 remaining options on day 2, 5 on day 3, etc

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- $\underset{n}{7}$ hats, $\underset{k}{3}$ days?

  7 options on day 1, 6 remaining options on day 2, 5 on day 3, ~~etc~~

$$7\cdot6\cdot5\cdot\cancel{4\cdot3\cdot2\cdot1} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1} = \frac{7!}{4!} = \frac{7!}{(7-3)!} = \frac{n!}{(n-k)!}$$

Notation: $(7)_3 = (n)_k = P(n,k)$ $\boxed{\text{permutation}}$

n items, k choices : repetition <u>not</u> allowed, order matters

- 12 people, must choose (i) president (ii) vice president (iii) treasurer

$\underset{n}{12}$ items, $\underset{k}{3}$ choices – each person can hold only one position

$$12\cdot 11\cdot 10 = \frac{12!}{(12-3)!} = \underset{\substack{0\\1}}{\overset{(n-0)}{n}}\cdot\underset{\substack{1\\2}}{(n-1)}\cdot\underset{\substack{2\\3}}{(n-2)}\cdot\underset{\substack{\cdots\\\cdots}}{\cdots}\cdot\underset{\substack{k-1\\k}}{(n-(k-1))}$$

$$n! = (n)_n = P(n,n)$$

$n>0:$ $(n)_0 = P(n,0) = 1$ 1 way to choose 0 items from n

$0! = (0)_0 = P(0,0) = 1$ 1 way to choose 0 items from 0

definition

it helps to think of lists or sets:

how many ways can we form an ordered list of 0 items?

↳ 1 way: the empty list

for $n \geq 1$: $n! = \prod_{i=1}^{n} i = n \cdot (n-1)!$ thus for $n=1$, $1! = 1 \cdot 0! = 1$

n! is related to choosing without repetition,
but also to rearranging:

- how many ways to rearrange (i.e. permute) the letters in "WORD"?

1 2 3 4

4!

will revisit this

n items to choose from, k choices to make

does order matter?

is repetition allowed?

	YES	NO
YES	n^k	?
NO	$\frac{n!}{(n-k)!}$ $P(n,k)$	?

combination without replacement

n items, k choices: repetition not allowed, order doesn't matter

- how many ways can we form a committee of 3, from 12 people?
  - ↳ all duties (tasks) shared, but we need 3 distinct people
  - ↳ not the same as: (i) president (ii) vice president (iii) treasurer & not n^k

- lets assign order* temporarily

solution*: $P(12,3) = \frac{12!}{(12-3)!}$

- for any triple of people A, B, C that might be chosen:
  - ↳ could get (A,B,C) (A,C,B) (B,A,C) (B,C,A) (C,A,B) (C,B,A) 3! ways

- solution: eliminate overcounting: $\frac{12!}{(12-3)!\cdot 3!}$

"n choose k"

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\cdot k!} = \binom{n}{n-k}$$

how many ways to leave 9 people out?

n items, k choices : repetition not allowed, order doesn't matter

- how many 8-bit binary numbers have exactly 3 1's ?
  n k

11100000, 01110000, 01010001 etc

↳ like committee

$$\binom{8}{3} = \frac{8!}{(8-3)!\cdot 3!} = 56$$

n items, k choices: repetition not allowed, order doesn't matter

$$C(n,1) = \binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n \qquad = P(n,1)$$

$$C(n,0) = \binom{n}{0} = \frac{n!}{(n-0)!\,0!} = 1 \qquad = P(n,0)$$

$$C(n,n) = \binom{n}{n} = \frac{n!}{(n-n)!\,n!} = 1 \qquad \neq P(n,n)$$

$$C(n,2) = \binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

P(n,2) double-counts

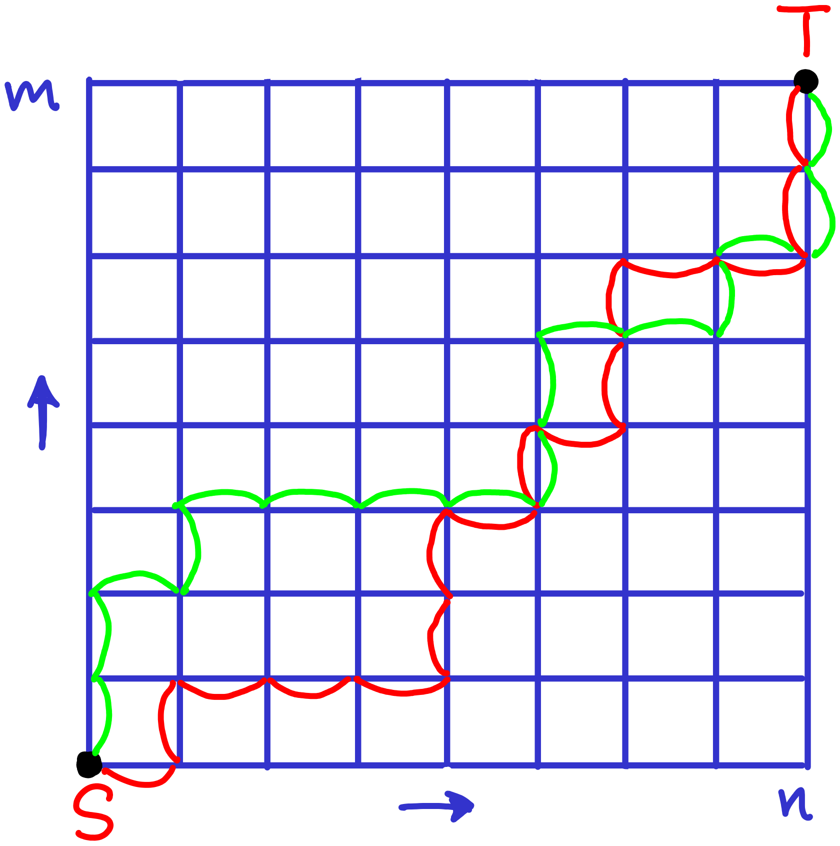
$$C(n,n+1) = \binom{n}{n+1} = 0 \qquad = \binom{n}{-1}$$

How many ways are there to get from $S=(0,0)$ to $T=(n,m)$ if you always take a step up or to the right?

- $S \to T$ must have $n+m$ steps.
- m of these steps go up.
- We must choose when to go up.

~ binary number with length $n+m$ and exactly m 1's.

$$\binom{n+m}{m} = \frac{(n+m)!}{((n+m)-m)!\,m!} = \frac{(n+m)!}{n!\,m!}$$



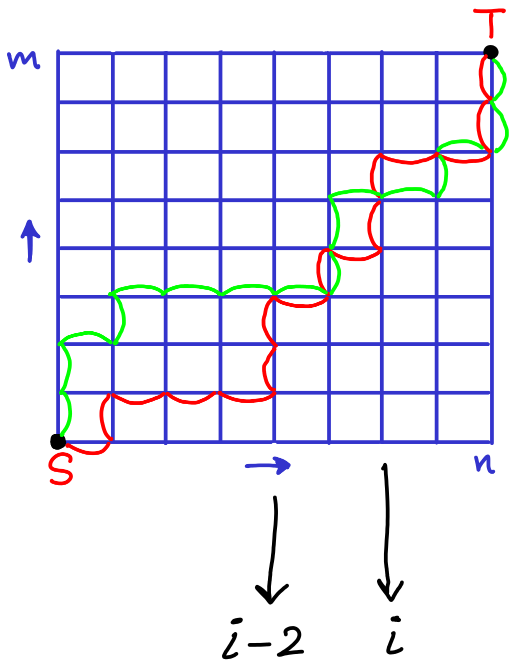
CONTEXT

repetition allowed? NO   order matters? NO

~~it looks like:~~

- ~~we can (must) move up repeatedly~~
- ~~a move up at position i comes after a move up at position < i~~

These are out of context



There are n+m steps

- we can't repeat a move up at the same step
- we care only about the set of steps where we move up.

n items to choose from, k choices to make

	does order matter?	
is repetition allowed?	YES	NO
YES	n^k	?
NO	$\frac{n!}{(n-k)!}$ $P(n,k)$	$\frac{n!}{(n-k)!k!}$ $C(n,k)$

combination with replacement

n items, k choices : repetition allowed, order doesn't matter

- $\underset{n}{5}$ types of donuts $d_1\ \ d_2\ \ d_3\ \ d_4\ \ d_5$ unlimited supply
- how many ways can you choose $\overset{k}{12}$ donuts ? $\boxed{\text{"n multichoose k"}}$

e.g. $\underset{d_1}{00000}\ |\ \underset{d_2}{}\ |\ \underset{d_3}{00}\ |\ \underset{d_4}{0}\ |\ \underset{d_5}{0000}$

↪ encoding: need $\underset{k}{12}$ 0's, include $\underset{n-1}{4}$ borders (1's)

- how many $\underset{k+(n-1)}{16}$-bit binary numbers have exactly $\underset{n-1}{4}$ 1's ? $\binom{16}{4}$

$$\left(\!\!\binom{n}{k}\!\!\right) = C(k+n-1,\ n-1) = \binom{k+n-1}{n-1} = C(k+n-1,\ k) = \binom{k+n-1}{k}$$

n items, k choices : repetition allowed, order doesn't matter

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$\binom{k+n-1}{n-1}$

committees instead of donuts :

- form a committee of k people, from n countries
  (each country has >k people, no restrictions on diversity)
- how many ways could countries be represented?

"what is the size of the set of country representations?"

(in a committee of k people, from n countries)

* in a committee of 0 people, from 0 countries? → 1

empty set

n items, k choices: repetition allowed, order doesn't matter

	$\binom{k+n-1}{n-1}$	$\binom{k+n-1}{k}$
• 1 type of donut, choose $k \geq 1$ donuts	$\binom{k+1-1}{1-1} = \binom{k}{0} = \frac{k!}{(k-0!)0!} = 1$	$\binom{k+1-1}{k}$
• 0 type of donut, choose $k \geq 1$ donuts	$\binom{k+0-1}{0-1} = \binom{k-1}{-1} = 0$	$\binom{k+0-1}{k}$
• $n \geq 1$ types of donuts, choose 0 donuts	$\binom{0+n-1}{n-1} = \frac{(n-1)!}{(n-1-0)!0!} = 1$	$\binom{0+n-1}{0}$
• 0 type of donut, choose 0 donuts	$\binom{0+0-1}{0-1} = \binom{-1}{-1}$ requires extended definition of C(n,k)	$\binom{0+0-1}{0} = 1$

n items to choose from, k choices to make

does order matter?

		YES	NO
is repetition allowed?	YES	n^k	$\binom{k+n-1}{n-1}$ $\left(\!\binom{n}{k}\!\right) = C(k+n-1, n-1)$
	NO	$\frac{n!}{(n-k)!}$ $P(n,k)$	$\frac{n!}{(n-k)!\,k!}$ $\binom{n}{k} = C(n,k)$