$$(x+y)^3 = \underbrace{(x+y)}\cdot\underbrace{(x+y)}\cdot\underbrace{(x+y)}_{}$$

$n=2$ items to choose from $\qquad$ $k=3$ choices $\qquad$ $n^k = 2^3 = 8$ ways

| $xxx$ | + | $xxy$ | + | $xyx$ | + | $xyy$ | + | $yxx$ | + | $yxy$ | + | $yyx$ | + | $yyy$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 000 | | 001 | | 010 | | 011 | | 100 | | 101 | | 110 | | 111 |
| 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 |

how many ways to get

$x^0y^3$ ? $\binom{3}{3} = 1$

$x^1y^2$ ? $\binom{3}{2} = 3$ choose two y's from three (1's)

$x^2y^1$ ? $\binom{3}{1} = 3$

$x^3y^0$ ? $\binom{3}{0} = 1$

$$(x+y)^n = (x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdots$$

- $2^n$ terms (including duplicates) e.g. $\left.\begin{matrix} xxyyx \\ yxyxy \end{matrix}\right\} x^3y^2$

- each term has $n$ $x$'s & $y$'s (combined) : $x^{n-k}\cdot y^k$ $(0 \le k \le n)$

- how many ways to get $x^{n-k}\cdot y^k$? $\left\{\begin{matrix} \text{choose } k \text{ } y\text{'s from } n \\ \binom{n}{k} \end{matrix}\right.$

# Binomial theorem

$x+y$ : binomial

(polynomial with 2 terms)

binomial coefficient

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k}$$

binomial expression

(symmetry)

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k} \cdot y^{n-k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k$$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot 1^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

number of binary numbers

number of ways to have precisely

$$\left\{ \begin{array}{l} \text{no } 1\text{'s}, \\ + \text{ one } 1, \\ \quad \vdots \\ + \ n-1 \ 1\text{'s}, \\ + \ n \ 1\text{'s} \end{array} \right\}$$

in a string with $n$ digits

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k$$

$k=0$ $\quad$ $k=1$ $\quad$ $k=2$ $\quad$ $k=3$ $\quad$ $k=4$ $\quad$ $k=5$ $\quad$ $k=6$

$n=0$ $\quad$ $1x^0 \cdot y^0$

$n=1$ $\quad$ $1x^1 \cdot y^0 + 1x^0 \cdot y^1$

$n=2$ $\quad$ $1x^2 \cdot y^0 + 2x^1 \cdot y^1 + 1x^0 \cdot y^2$

$n=3$ $\quad$ $1x^3y^0 + 3x^2y^1 + 3x^1y^2 + 1x^0y^3$

$n=4$ $\quad$ $1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4$

$n=5$ $\quad$ $1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^0y^5$

$n=6$ $\quad$ $1x^6y^0 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + 1x^0y^6$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k$$

| | k=0 | k=1 | k=2 | k=3 | k=4 | k=5 | k=6 |
|---|---|---|---|---|---|---|---|
| n = 0 | 1 | | | | | | |
| n = 1 | 1 | 1 | | | | | |
| n = 2 | 1 | 2 | 1 | | | | |
| n = 3 | 1 | 3 | 3 | 1 | | | |
| n = 4 | 1 | 4 | 6 | 4 | 1 | | |
| n = 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| n = 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k$$

$k=0 \quad k=1 \quad k=2 \quad k=3 \quad k=4 \quad k=5 \quad k=6$

$n=0 \qquad \binom{0}{0}$

$n=1 \qquad \binom{1}{0} \quad \binom{1}{1}$

$n=2 \qquad \binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$

$n=3 \qquad \binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$

$n=4 \qquad \binom{4}{0} \quad \binom{n-1}{k-1} \quad \binom{n-1}{k} \quad \binom{4}{3} \quad \binom{4}{4}$

$$\binom{n-1}{k-1} + \binom{n-1}{k} \to \binom{n}{k}$$

$n=5 \qquad \binom{5}{0} \quad \binom{5}{1} \quad \binom{n}{k} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$

$n=6 \qquad \binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$

# Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\begin{tikzcd}
\binom{n-1}{k-1} \arrow[dr] & + & \binom{n-1}{k} \arrow[dl] \\
 & \binom{n}{k} &
\end{tikzcd}$$

$$\left.\begin{matrix} k = 0 \\ n = k \end{matrix}\right\} \text{base cases}$$

$$\boxed{\begin{matrix} k \neq 0 \\ n-k \neq 0 \end{matrix}}$$

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(n-1-(k-1))! \cdot (k-1)!} + \frac{(n-1)!}{(n-1-k)! \cdot k!}$$

$$= \frac{(n-1)!}{(n-k)! \cdot (k-1)!} + \frac{(n-1)!}{(n-1-k)! \cdot k!}$$

$$= \frac{(n-1)! \cdot k}{(n-k)! \cdot (k-1)! \cdot k} + \frac{(n-1)! \cdot (n-k)}{(n-1-k)! \cdot k! \cdot (n-k)}$$

$$= \frac{(n-1)! \cdot k}{(n-k)! \cdot k!} + \frac{(n-1)! \cdot (n-k)}{(n-k)! \cdot k!}$$

$$= \frac{(n-1)! \cdot n}{(n-k)! \cdot k!} = \frac{n!}{(n-k)! \cdot k!} = \binom{n}{k}$$

# Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

sometimes written as

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\left.\begin{array}{l} k=0 \\ n=k \end{array}\right\} \text{base cases}$$

$k=0$, $k=1$, $k=2$, $k=3$, $k=4$, $k=5$, $k=6$

$n=0$: $\binom{0}{0}$

$n=1$: $\binom{1}{0}$ $\binom{1}{1}$

$n=2$: $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$n=3$: $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$n=4$: $\binom{4}{0}$ $\binom{n-1}{k-1}$ $+$ $\binom{n-1}{k}$ $\binom{4}{3}$ $\binom{4}{4}$

$n=5$: $\binom{5}{0}$ $\binom{5}{1}$ $\binom{n}{k}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

$n=6$: $\binom{6}{0}$ $\binom{6}{1}$ $\binom{6}{2}$ $\binom{6}{3}$ $\binom{6}{4}$ $\binom{6}{5}$ $\binom{6}{6}$

## Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Alternate proof $\longrightarrow$

- You are one of $n$ people.
- How many ways to form a committee of size $k$? $\Big\}$ $\binom{n}{k}$

* precisely 2 disjoint scenarios:

You are on the committee OR You are off the committee

either way, $n-1$ people remain

choose $k-1$ remaining — must still choose $k$

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

base cases (exceptions)*
($\cancel{\exists}$ 2 scenarios)

$k=0$ $\cancel{\exists}$ committee (you're off)

$n=k$ all on committee