

$$(x+y)^3 = \underbrace{(x+y)}_{k=3 \text{ choices}} \cdot \underbrace{(x+y)}_{k=3 \text{ choices}} \cdot \underbrace{(x+y)}_{k=3 \text{ choices}}$$

$$n^k = 2^3 = 8 \text{ ways}$$

$n=2$ items to choose from

xxx	+	xxY	+	XYX	+	XYy	+	YXX	+	Yxy	+	YYX	+	YYY
000		001		010		011		100		101		110		111
0		1		2		3		4		5		6		7

how many ways to get

{	$x^0 y^3$?	$\binom{3}{3} = 1$	choose two y's from three (1's)
	$x^1 y^2$?	$\binom{3}{2} = 3$	
	$x^2 y^1$?	$\binom{3}{1} = 3$	
	$x^3 y^0$?	$\binom{3}{0} = 1$	

$$(x+y)^n = (x+y) \cdot (x+y) \cdot (x+y) \cdot (x+y) \cdot (x+y) \dots$$

- 2^n terms (including duplicates) e.g. $\begin{matrix} x & x & y & y & x \\ y & x & y & x & y \end{matrix} \} x^3 y^2$
- each term has n x 's & y 's (combined) : $x^{n-k} \cdot y^k$ ($0 \leq k \leq n$)
- how many ways to get $x^{n-k} \cdot y^k$? $\left\{ \begin{array}{l} \text{choose } k \text{ } y\text{'s from } n \\ \binom{n}{k} \end{array} \right.$

Binomial theorem

$x+y$: binomial
(polynomial with 2 terms)

binomial coefficient

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} \cdot y^k$$

binomial expression

$$= \sum_{k=0}^n \binom{n}{k} x^k \cdot y^{n-k}$$

(symmetry)

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} \cdot y^k$$

$$(1+1)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} \cdot 1^k$$

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

number of binary
numbers

number of ways
to have precisely

{

- no 1's,
- + one 1,
- ⋮
- + n-1 1's,
- + n 1's

}

in a string with n digits



$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} \cdot y^k$$

$n=0$ $1x^0 \cdot y^0$ $k=0$

$n=1$ $1x^1 \cdot y^0 + 1x^0 \cdot y^1$ $k=1$

$n=2$ $1x^2 \cdot y^0 + 2x^1 \cdot y^1 + 1x^0 \cdot y^2$ $k=2$

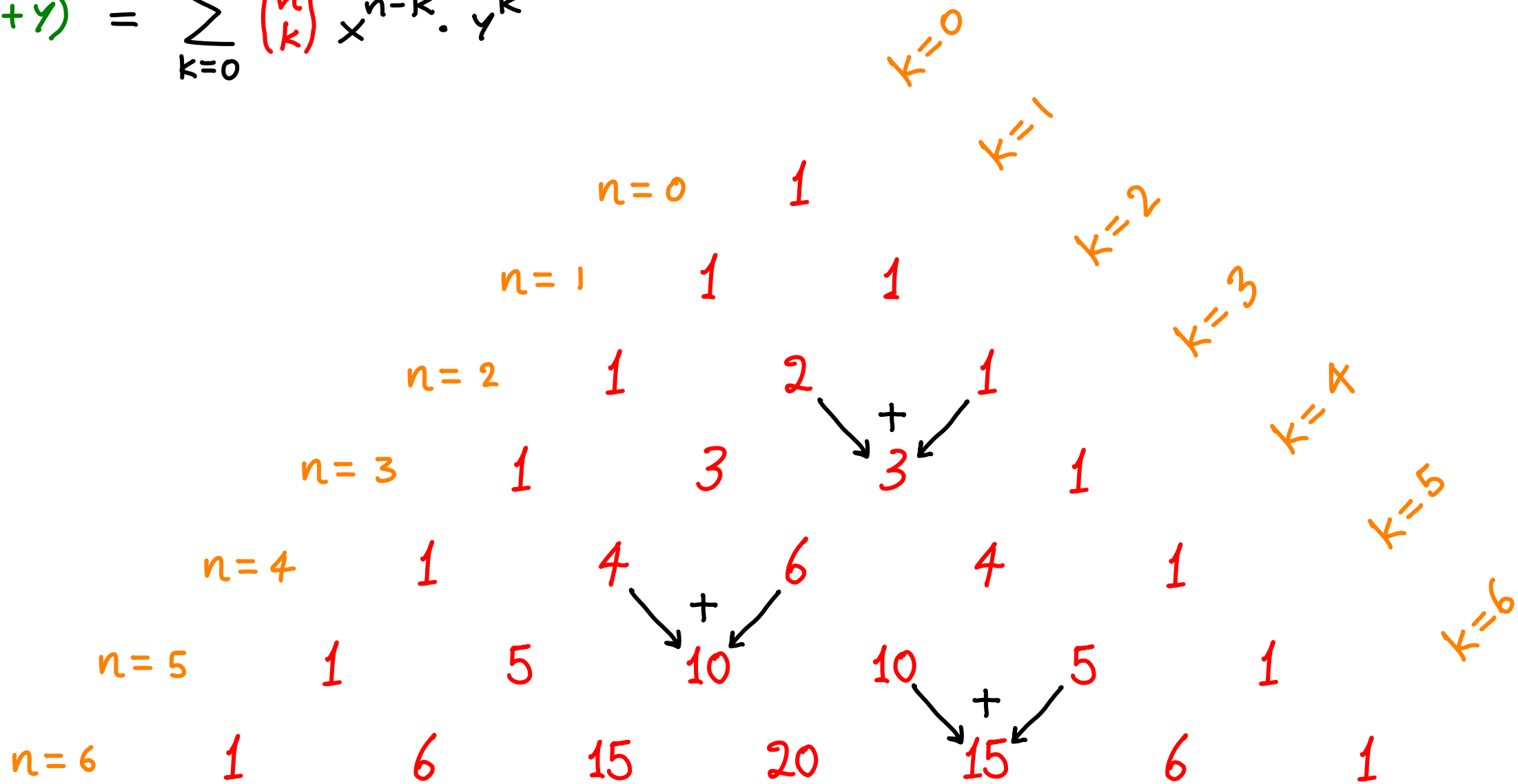
$n=3$ $1x^3 \cdot y^0 + 3x^2 \cdot y^1 + 3x^1 \cdot y^2 + 1x^0 \cdot y^3$ $k=3$

$n=4$ $1x^4 \cdot y^0 + 4x^3 \cdot y^1 + 6x^2 \cdot y^2 + 4x^1 \cdot y^3 + 1x^0 \cdot y^4$ $k=4$

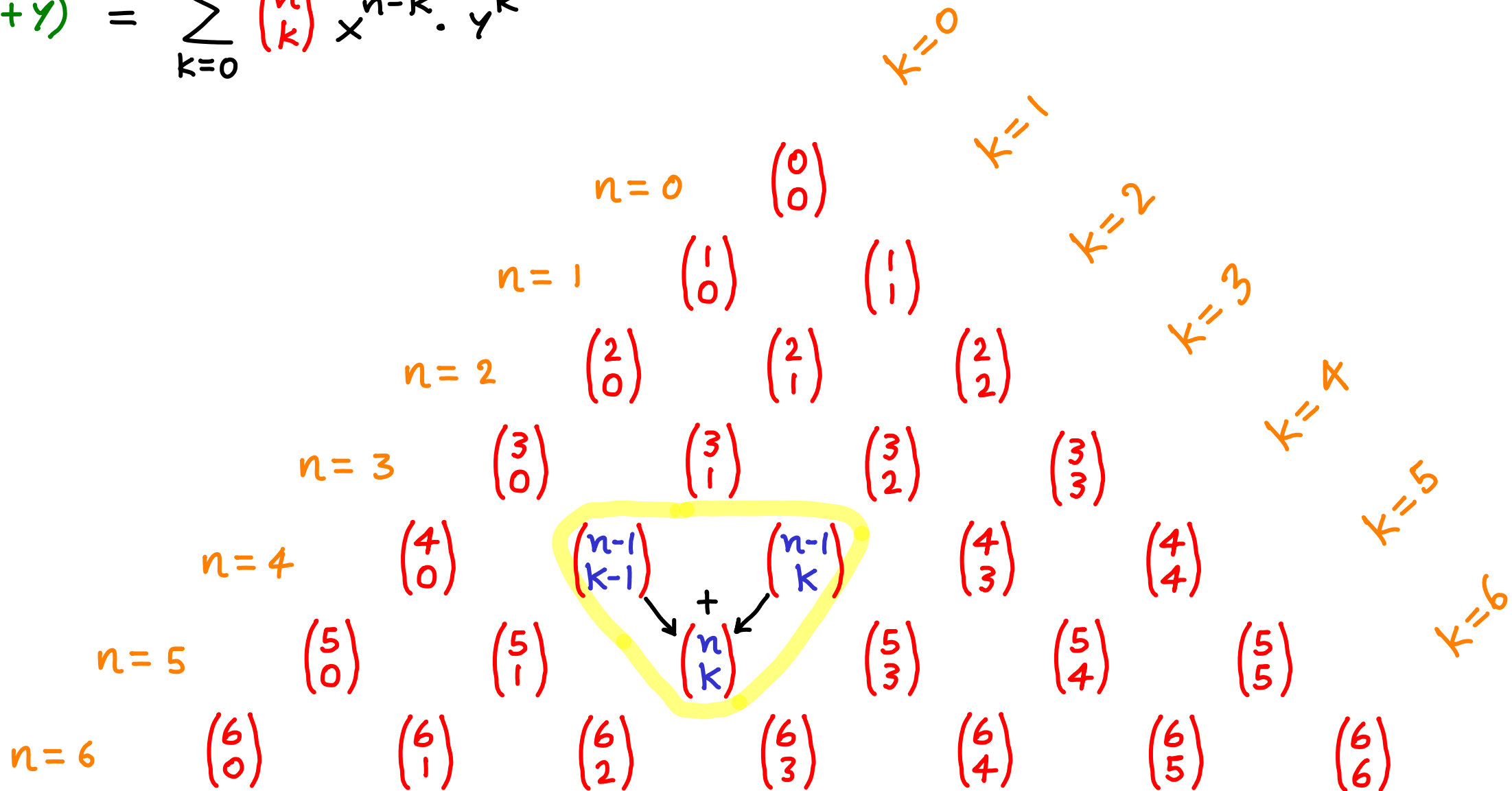
$n=5$ $1x^5 \cdot y^0 + 5x^4 \cdot y^1 + 10x^3 \cdot y^2 + 10x^2 \cdot y^3 + 5x^1 \cdot y^4 + 1x^0 \cdot y^5$ $k=5$

$n=6$ $1x^6 \cdot y^0 + 6x^5 \cdot y^1 + 15x^4 \cdot y^2 + 20x^3 \cdot y^3 + 15x^2 \cdot y^4 + 6x^1 \cdot y^5 + 1x^0 \cdot y^6$ $k=6$

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} \cdot y^k$$

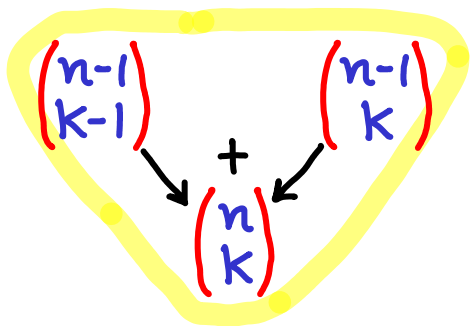


$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} \cdot y^k$$



Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$k=0$
 $n=k$ } base cases

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(n-1-(k-1))! \cdot (k-1)!} + \frac{(n-1)!}{(n-1-k)! \cdot k!}$$

$$= \frac{(n-1)!}{(n-k)! \cdot (k-1)!} + \frac{(n-1)!}{(n-1-k)! \cdot k!}$$

$$k \neq 0$$

$$n-k \neq 0$$

$$= \frac{(n-1)! \cdot k}{(n-k)! \cdot (k-1)! \cdot k} + \frac{(n-1)! \cdot (n-k)}{(n-1-k)! \cdot k! \cdot (n-k)}$$

$$= \frac{(n-1)! \cdot k}{(n-k)! \cdot k!} + \frac{(n-1)! \cdot (n-k)}{(n-k)! \cdot k!}$$

$$= \frac{(n-1)! \cdot n}{(n-k)! \cdot k!} = \frac{n!}{(n-k)! \cdot k!} = \binom{n}{k}$$

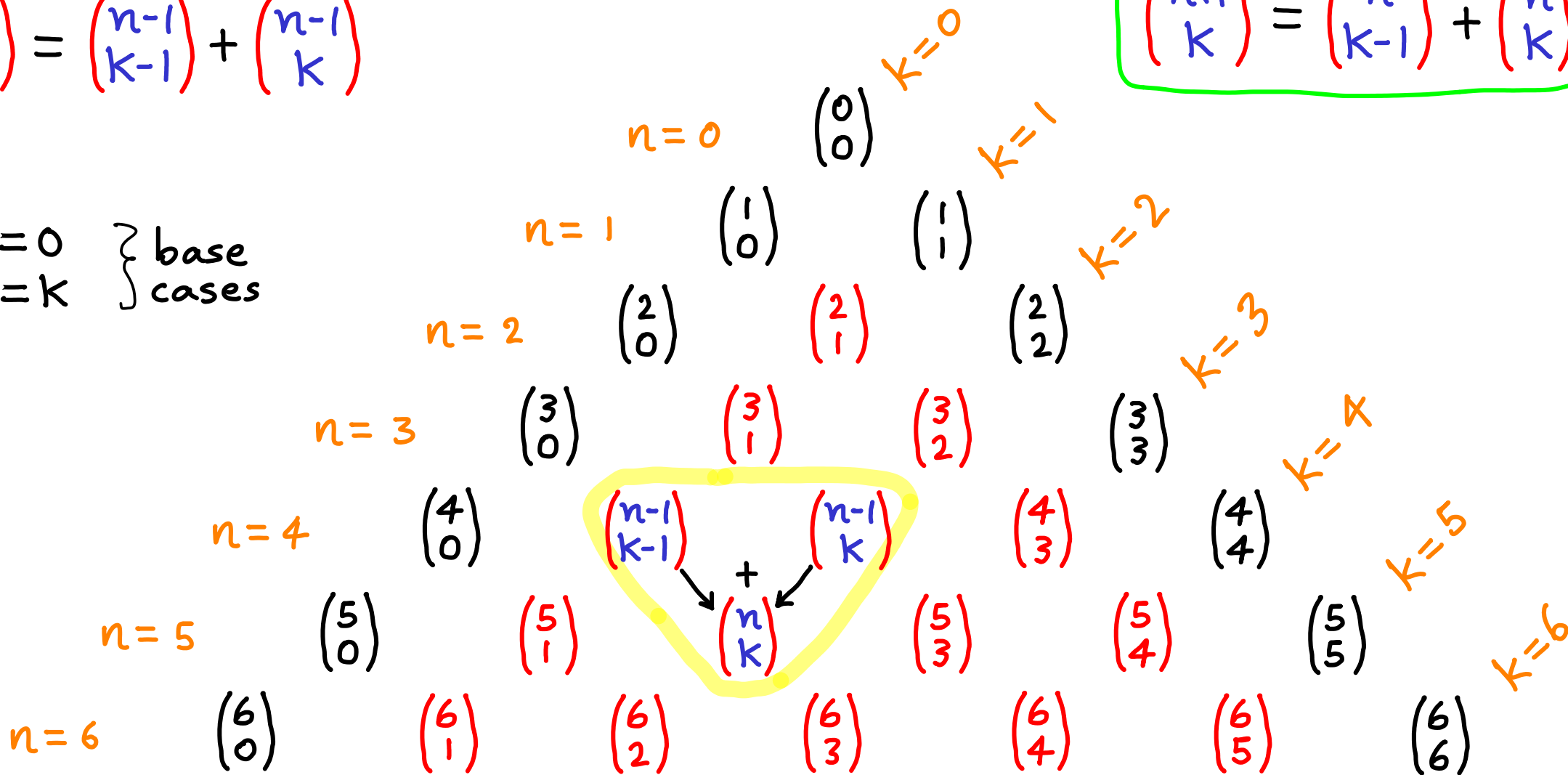
Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

sometimes written as

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$k=0$
 $n=k$ } base cases



Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Alternate proof $\rightarrow$

base cases (exceptions)*
($\nexists$ 2 scenarios)

$k=0$ $\nexists$ committee
(you're off)

$n=k$ all on committee

- You are one of n people.
- How many ways to form a committee of size k ? $\left. \vphantom{\begin{matrix} \text{How many ways to form} \\ \text{a committee of size } k? \end{matrix}} \right\} \binom{n}{k}$

* $\hookrightarrow$ precisely 2 disjoint scenarios:

You are on
the committee

OR

You are off
the committee

either way, $n-1$ people remain

$\swarrow$
choose $k-1$
remaining

$$\binom{n-1}{k-1}$$

+

$\searrow$
must still
choose k

$$\binom{n-1}{k}$$