dynamic BALANCED SEARCH TREES (NON-RANDOM)

Objectives : search , $\underbrace{\text{insert, delete}}$ in $O(\log n)$ time

↳ always maintain $\Theta(\log n)$ height & $\underbrace{\text{update}}$ in $O(\log n)$ time

RED-BLACK trees

Structure:

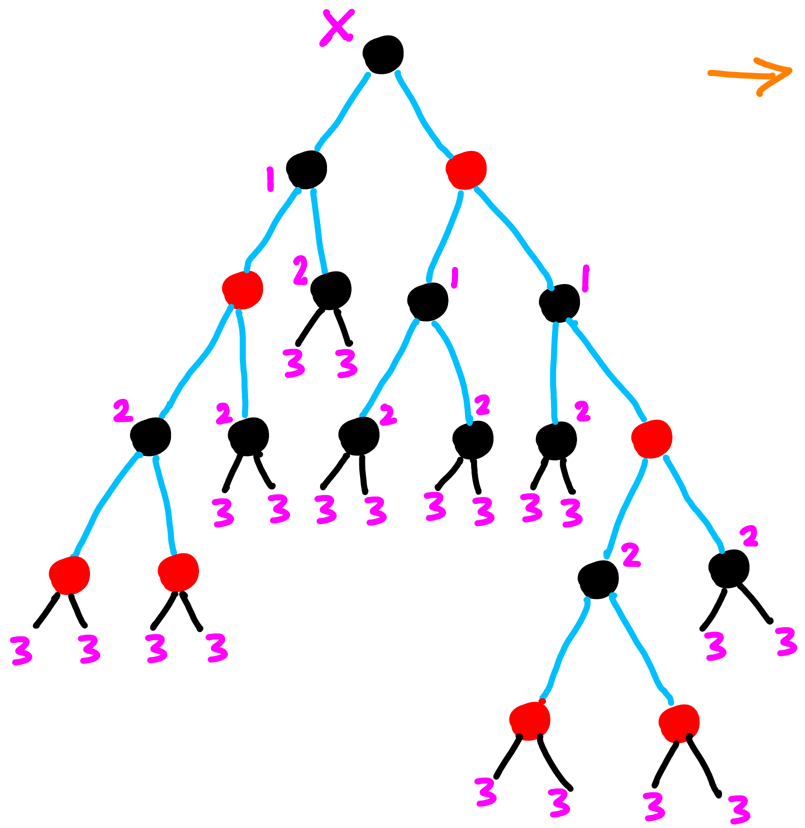
1) nodes are colored red or black.
2) root is always black.
3) add black "dummy" leaves so every "real" node has 2 children.

the important rules

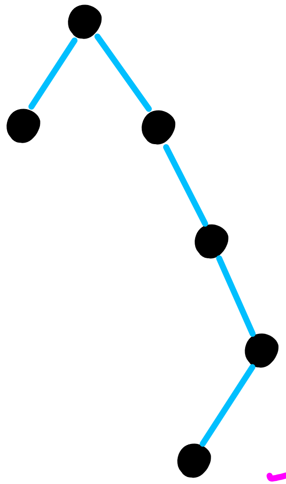
4) every red node has a black parent.
5) for any node x: all paths down to leaves contain equal number of black nodes $\underbrace{= \text{black-height}[x]}_{\text{not including } x}$

4) every red node has a black parent.

5) for any node x : all paths down to leaves contain equal number of black nodes = black-height[x]



→ Fails rule 5 ⇒ fix by making some nodes red.



No hope to recolor

...

too unbalanced

Any root→leaf path of size k must have $\geq \frac{k}{2}$ black nodes.

Rule 4

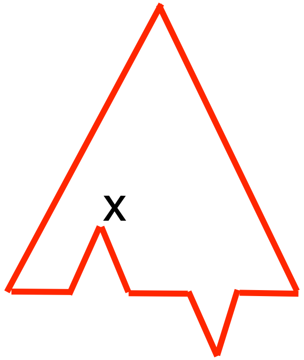
So if any path is >2 times longer than another, we can't make it RB.

Proof that RB trees have height < 2logn

1) Fact: If a tree is perfectly balanced the height is logn

2) If a tree is not perfectly balanced, there is a node, x, at depth d < logn

3) By our claim above there can't be any other node at depth > 2d



□

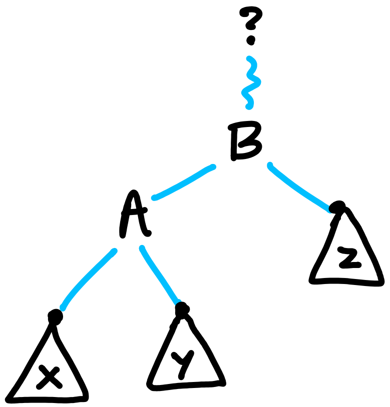
We have seen that RB trees are reasonably balanced : $\sim 2\log n$

↳ search, min, max, next, prev : $O(\log n)$ time.

Next: how to update RB trees (insert, delete)

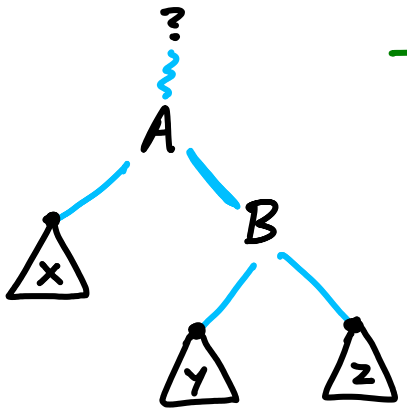
ROTATIONS in arbitrary BSTs

T_1



right-rotate(T_1, B) →

← left-rotate(T_2, A)



T_2

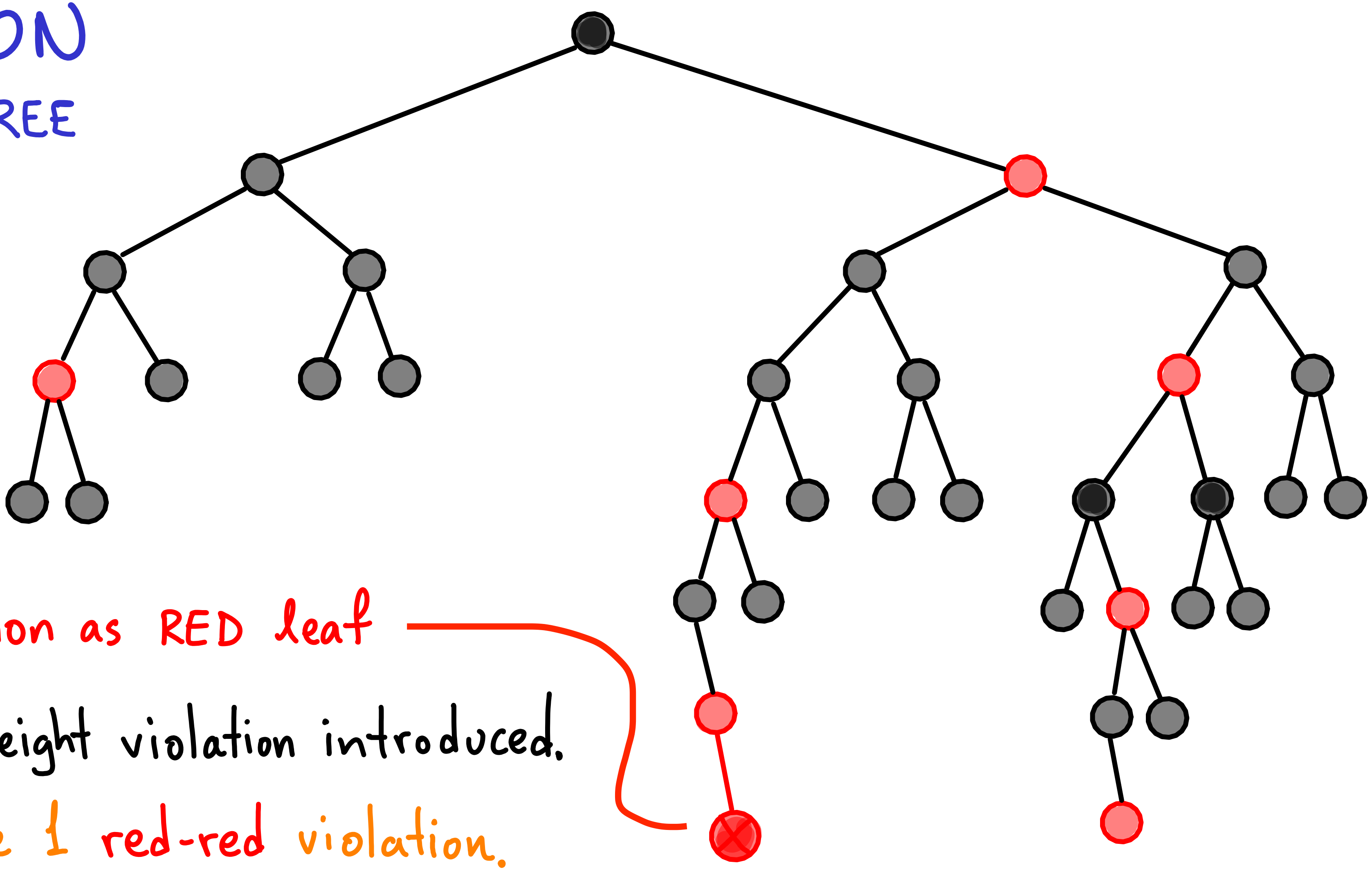
$x \leq A \leq y \leq B \leq z$

$O(1)$ time

$x \leq A \leq y \leq B \leq z$

INSERTION

IN A R-B TREE



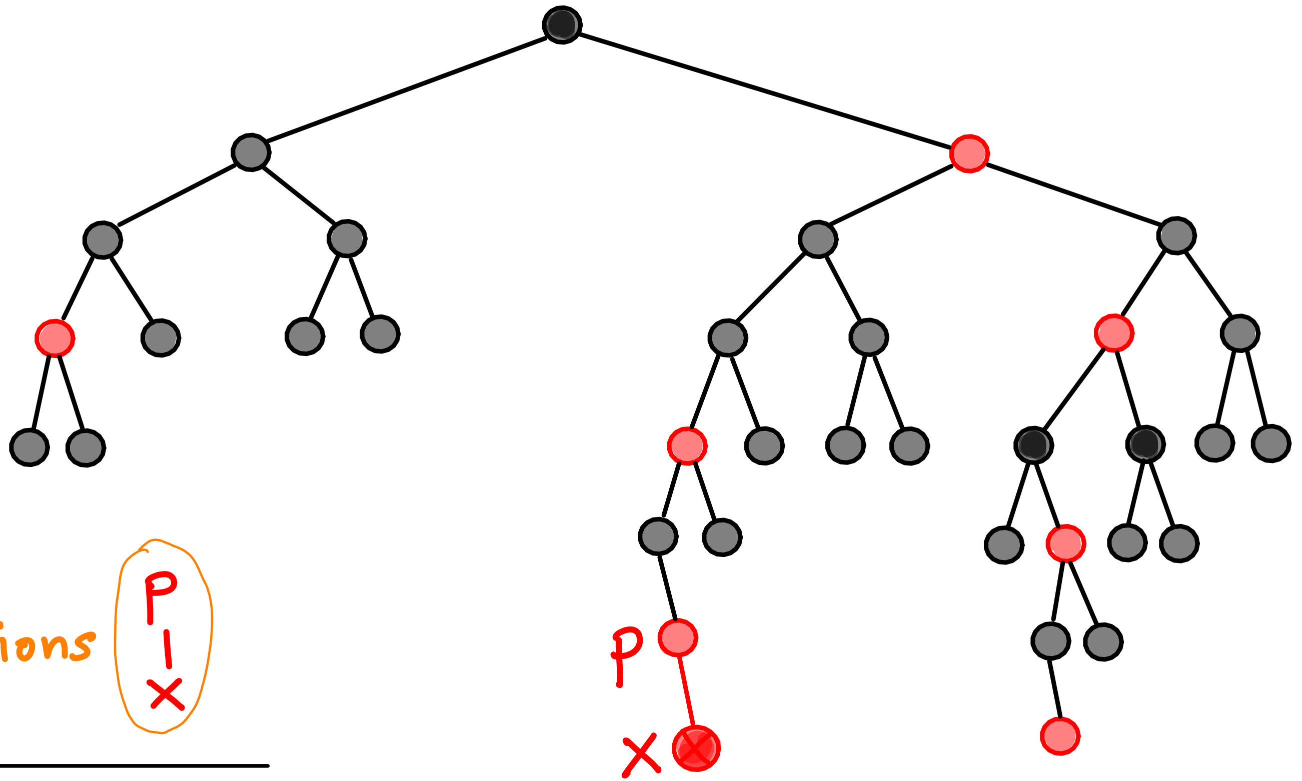
Regular insertion as RED leaf

↳ No black height violation introduced.

↳ Might have 1 red-red violation.

↳ If so, start fixing: locally & upward

fixing: locally & upward



Loop invariants:

- No black height violation.
- 1 red-red violation, at positions p | x

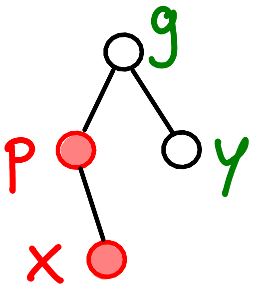
Loop outcomes:

Obtain R-B tree

OR

Redefine ancestor as x = fix upward

Local positions considered in loop



x & p : both red if we needed to continue loop

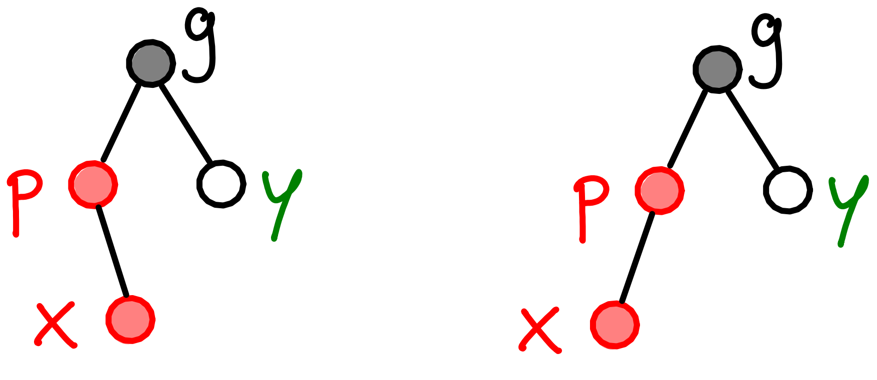
g: grandparent of x

y: "uncle" of x

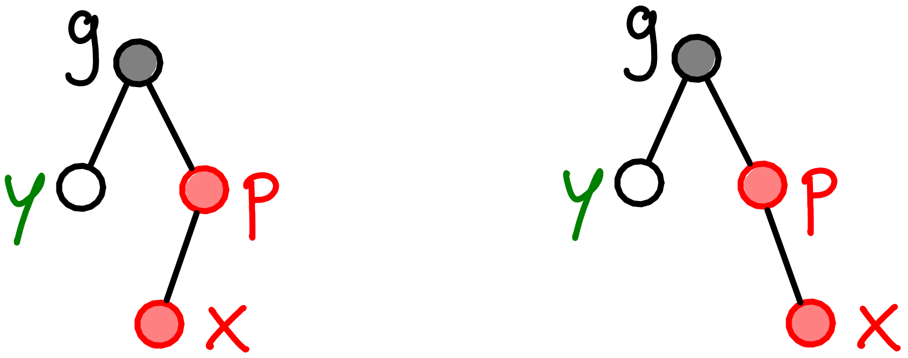
g must exist: must be black
because p can't be root

y must exist
(could be a fake leaf)

2 possible shapes...



...and their mirror images



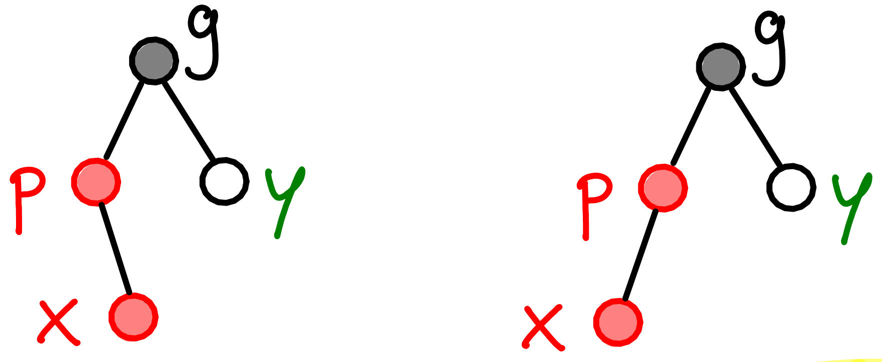
x & p : both red if we needed to continue loop

g: grandparent of x

y: "uncle" of x

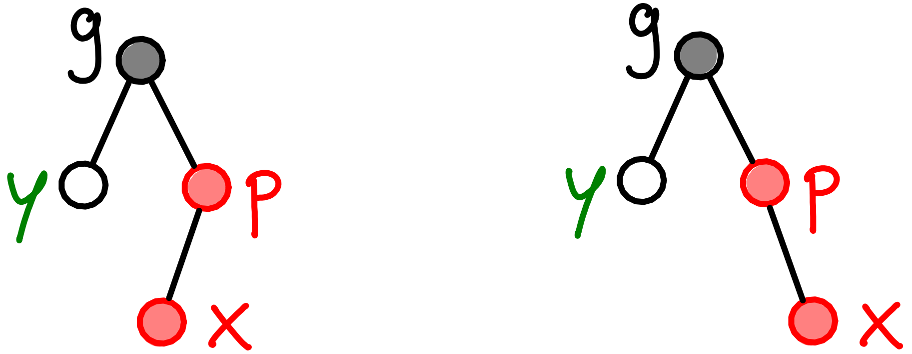
We will handle the 2 main shapes:

2 possible shapes...



...and their mirror images

The others can be done by symmetry:



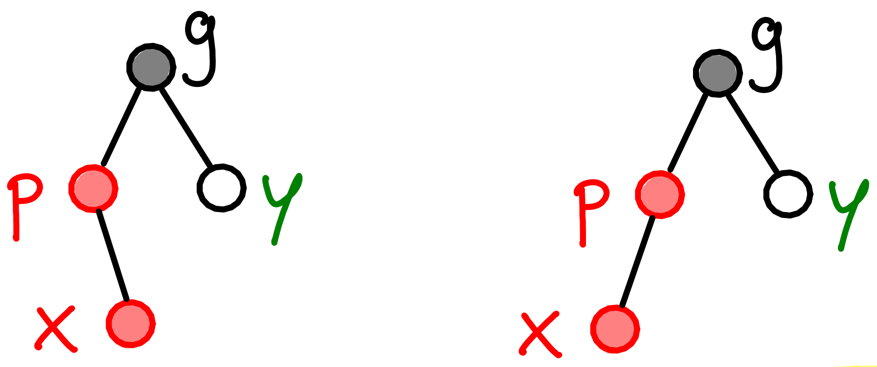
x & p : both red if we needed to continue loop

g: grandparent of x

y: "uncle" of x

We will handle the 2 main shapes:

2 possible shapes...



Case 1 : y is red

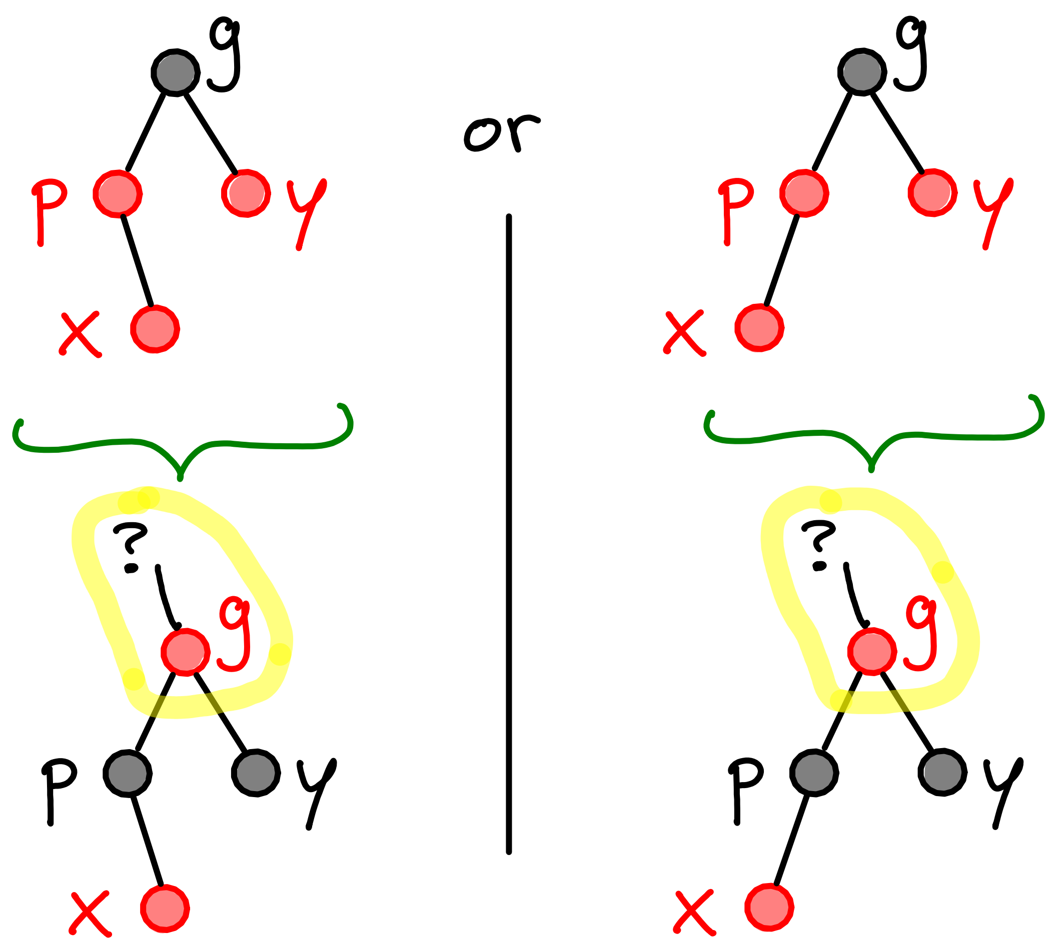
Cases 2 & 3 : y is black

Case 1 : y is red

↳ Recolor p, g, y

- Preserve black-height invariant.
  (total B nodes on any path from global root down to leaves)
- Eliminate p–x violation.
- If no new violation ? ↕ g then DONE

(note: if g = root color g black)



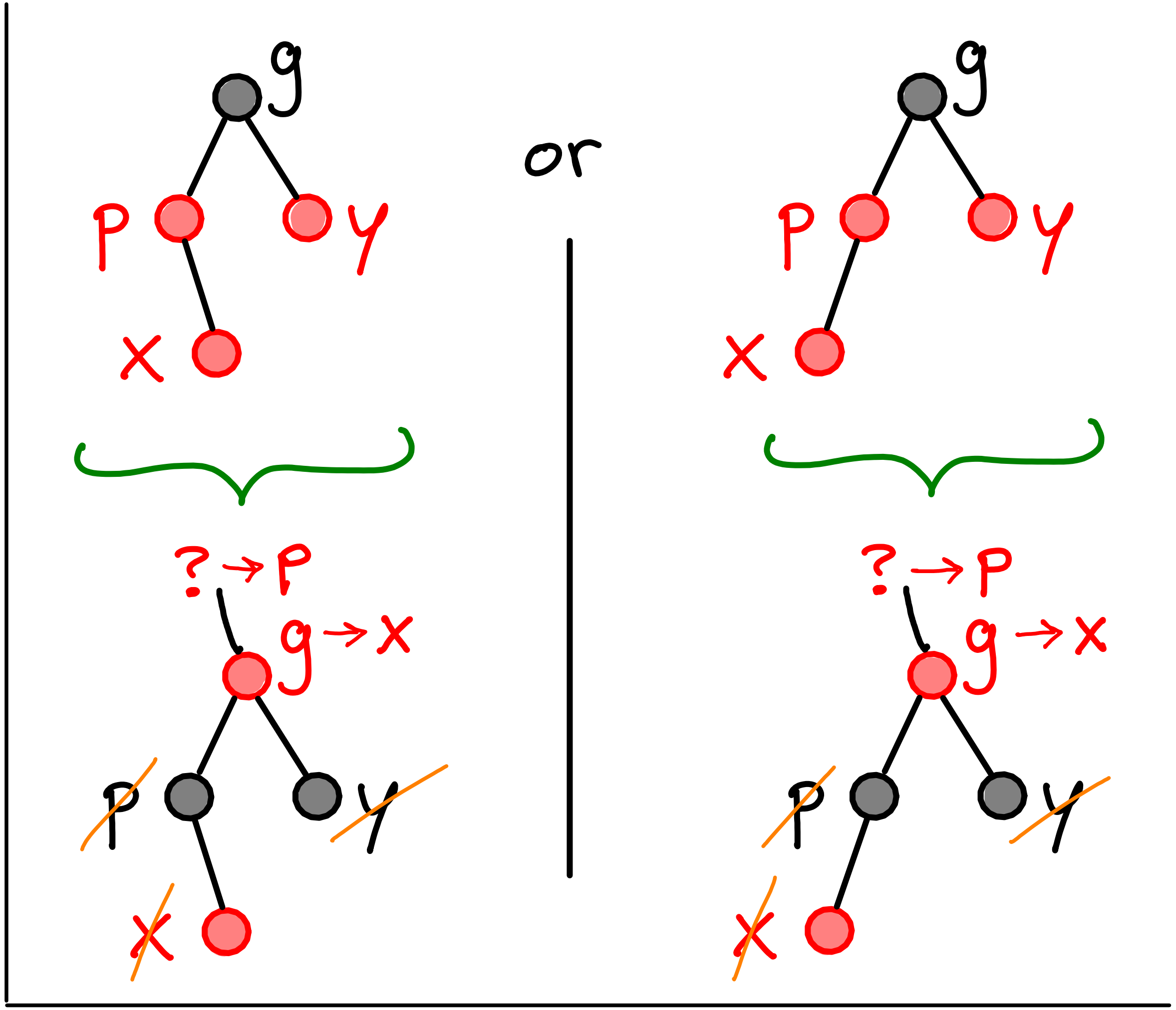
Case 1 : y is red

↳ Recolor p, g, y   O(1)

- Preserve black-height invariant. (total B nodes on any path from global root down to leaves)
- Eliminate $\begin{array}{c} p \\ | \\ x \end{array}$ violation.
- Might cause violation $\begin{array}{c} ? \\ \updownarrow \\ g \end{array}$

  otherwise DONE

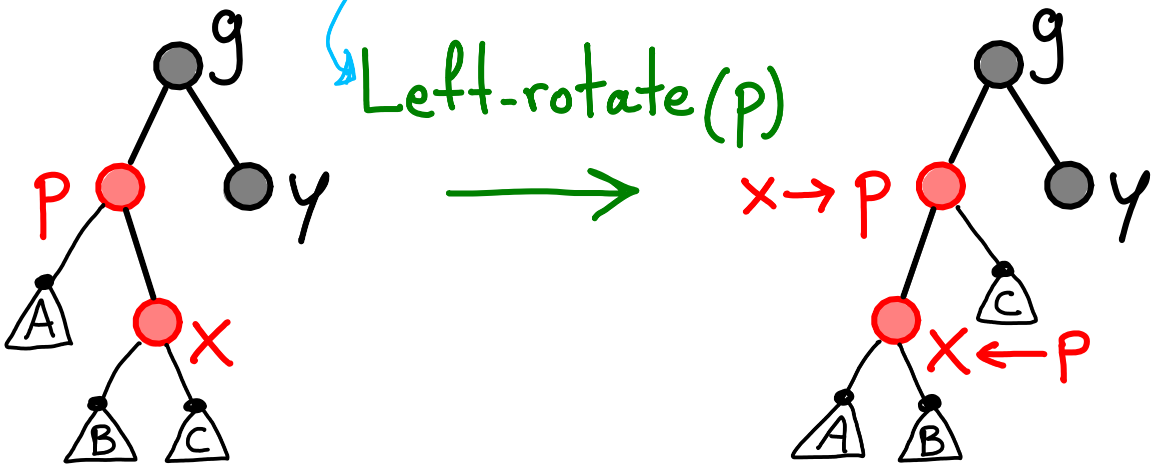
→ fix upward → ? becomes p, g becomes x → Invariants restored



Cases 2 & 3: y is black

- Preserve black-height invariant.
- Swap labels $x \leftrightarrow p$
- No new violation

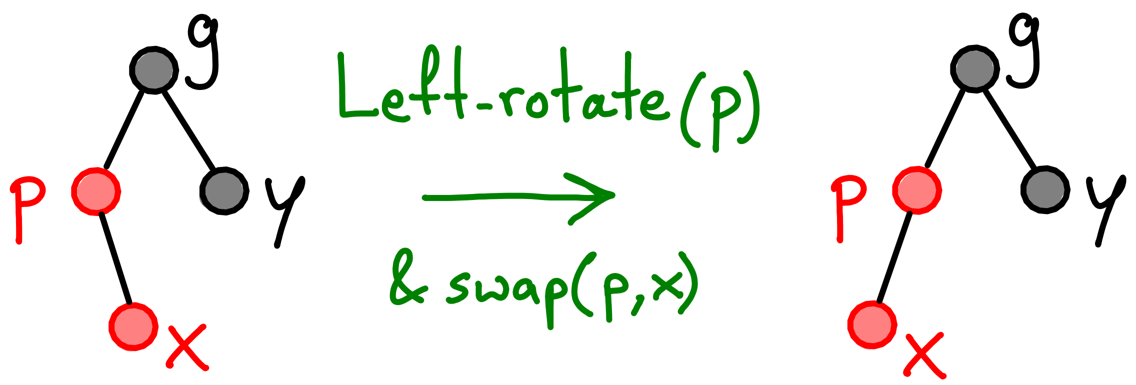
Case 2



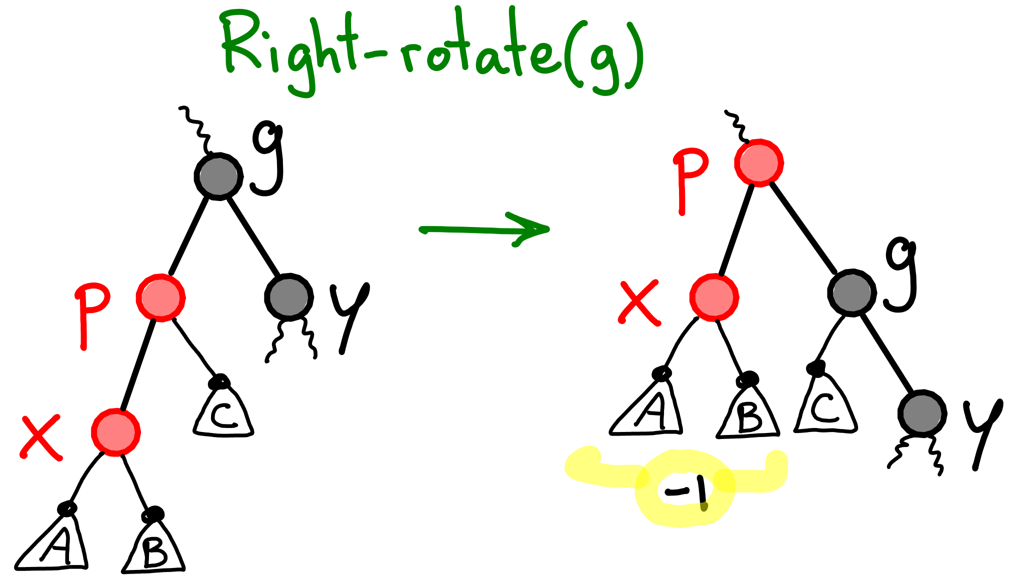
Cases 2 & 3: y is black

- ~~Preserve~~ black-height invariant.

Case 2 Case 3

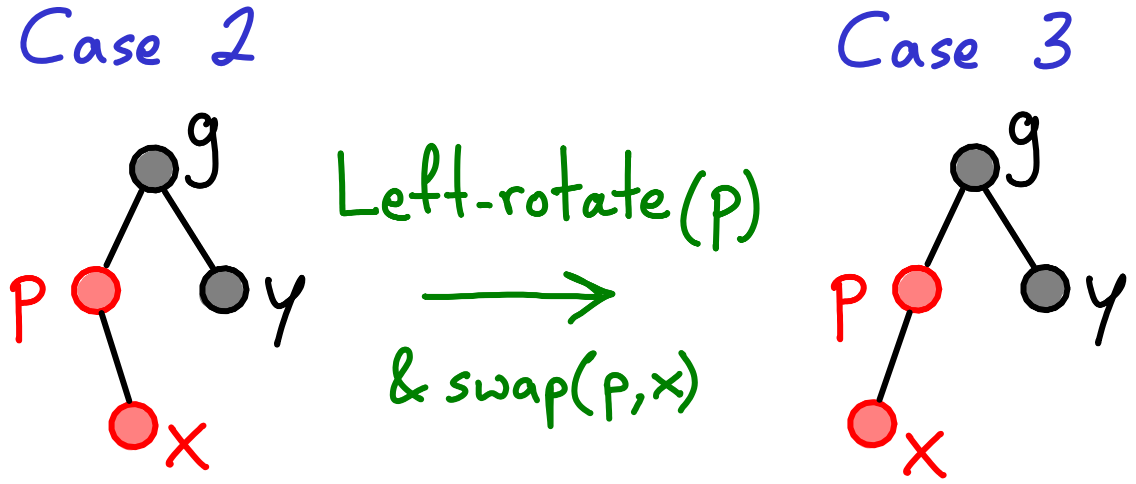


Handling Case 3:

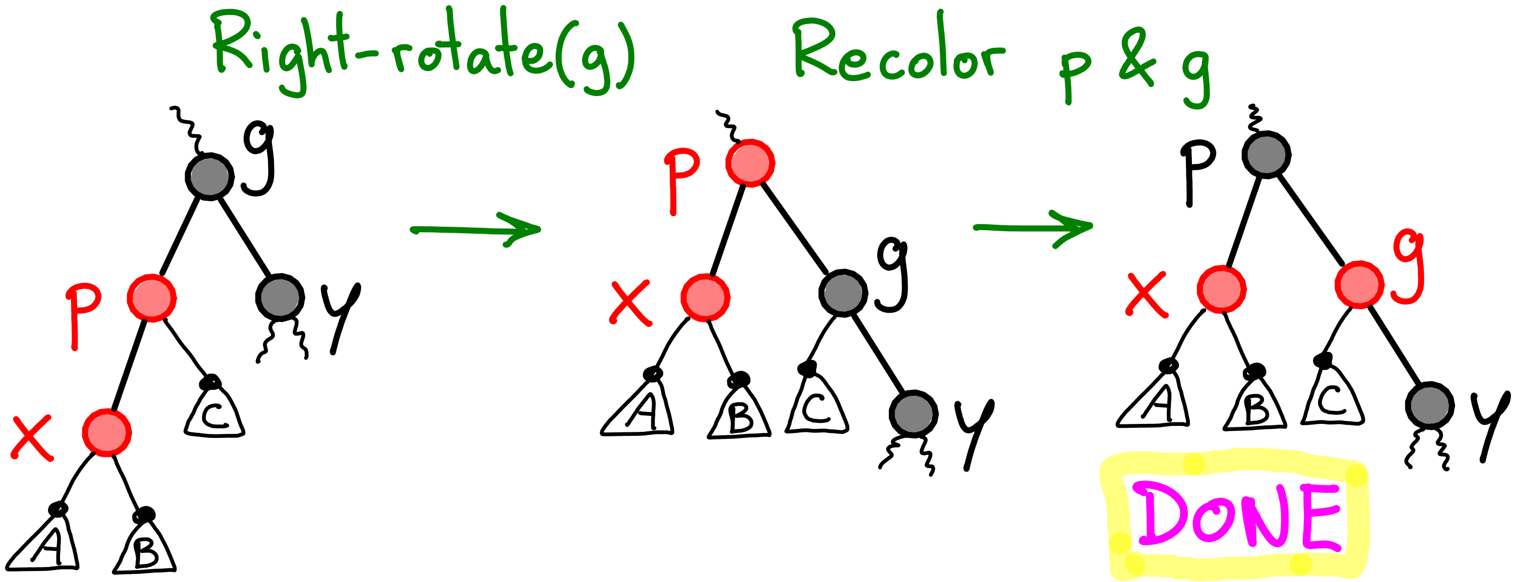


Cases 2 & 3: y is black

- Preserve black-height invariant.
- Eliminate p–x violation in case 3.
- No new violation



Handling Case 3:



Each case takes O(1) time

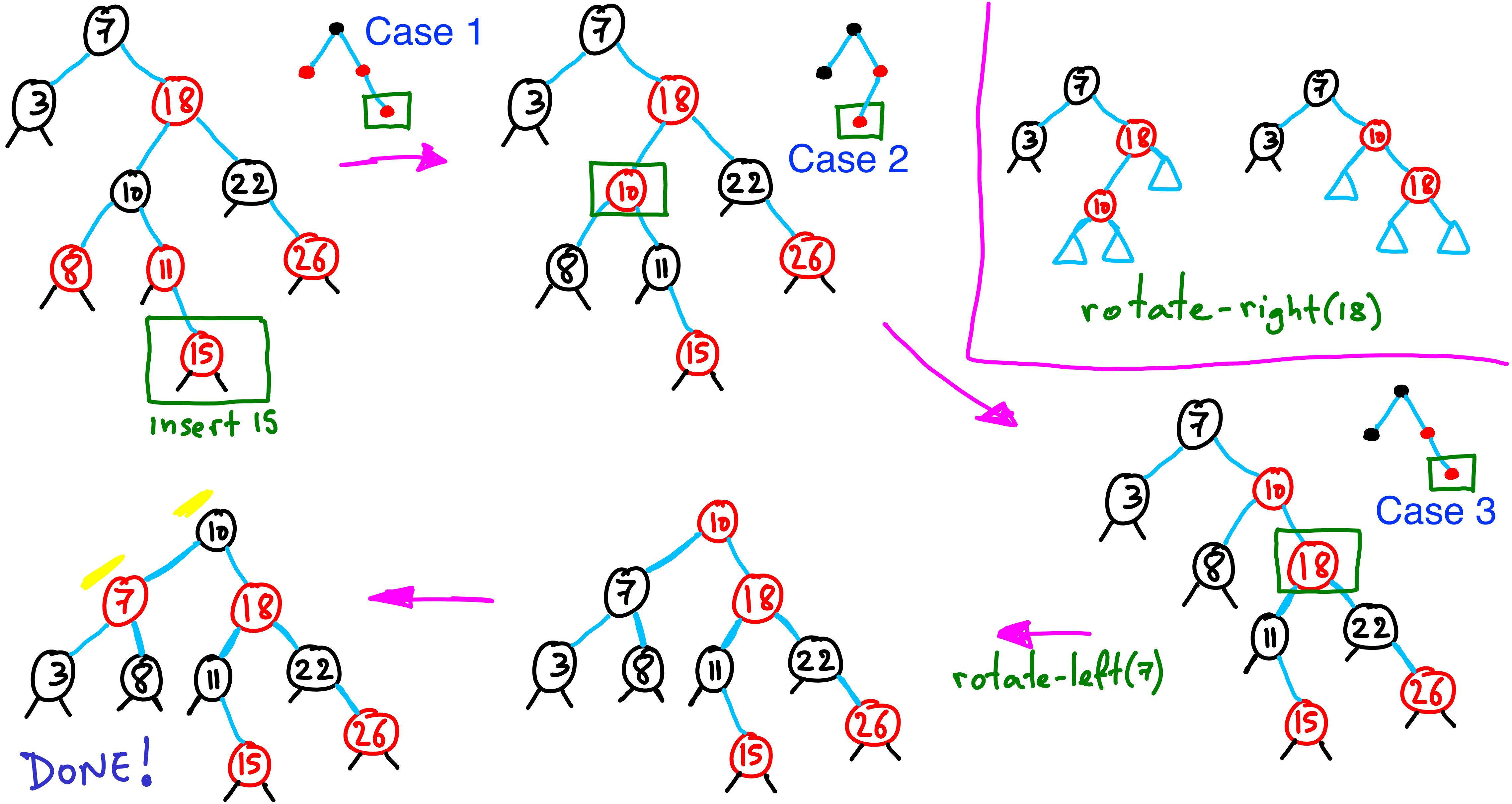
All together O(logn) time

Possibly lots of recoloring (Case 1)

followed by Case 3 OR Case 2 —> 3

(1 or 2 rotations, total)

* useful property



insert 15
Case 1
Case 2
rotate-right(18)
Case 3
rotate-left(7)
DONE!