PSET 2 solutions

1 [redacted] a) Let $\vec{a}(t) = \langle a_1(t), a_2(t), a_3(t) \rangle$

Let $\vec{b}(t) = \langle b_1(t), b_2(t), b_3(t) \rangle$

$$\vec{a}(t) \cdot \vec{b}(t) = a_1(t)b_1(t) + a_2(t)b_2(t) + a_3(t)b_3(t)$$

$$\frac{d}{dt}\left(\vec{a}(t) \cdot \vec{b}(t)\right) = a_1'(t)b_1(t) + a_1(t)b_1'(t) + a_2'(t)b_2(t) + a_2(t)b_2'(t) + a_3'(t)b_3(t) + a_3(t)b_3'(t)$$

$$= \langle a_1'(t), a_2'(t), a_3'(t) \rangle \cdot \langle b_1(t), b_2(t), b_3(t) \rangle + \langle a_1(t), a_2(t), a_3(t) \rangle \langle b_1'(t), b_2'(t), b_3'(t) \rangle$$

$$= \frac{d\vec{a}}{dt} \cdot \vec{b} + \vec{a} \cdot \frac{d\vec{b}}{dt}.$$

b) If speed is constant, then

$$0 = \frac{d}{dt} |\vec{V}(t)|^2$$

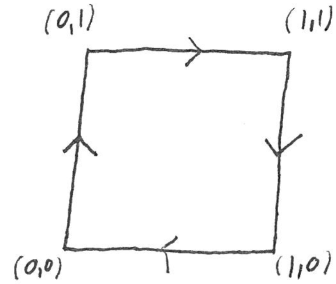
Recall that $\vec{V}(t) = \frac{d\vec{x}}{dt}$, and $|\vec{V}|^2 = \vec{V} \cdot \vec{V}$, $\vec{a} = \frac{d\vec{V}}{dt}$

So $0 = \frac{d}{dt}(\vec{V} \cdot \vec{V}) = \vec{V} \cdot \frac{d\vec{V}}{dt} + \frac{d\vec{V}}{dt} \cdot \vec{V}$ (by part a)

$$\Rightarrow \quad 0 = 2\vec{V} \cdot \vec{a}$$

So $\vec{V} \perp \vec{a}$

2 a)



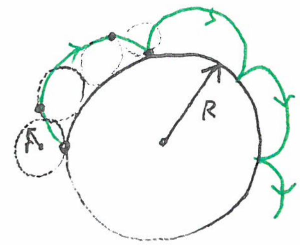
Let's build $\vec{X}(t)$ that traces each side in time 1.

$$\vec{X}(t) = \begin{cases} t\langle 0,1\rangle & 0 \le t \le 1 \\ \langle 0,1\rangle + (t-1)\langle 1,0\rangle & 1 \le t \le 2 \\ \langle 1,1\rangle + (t-2)\langle 0,-1\rangle & 2 \le t \le 3 \\ \langle 1,0\rangle + (t-3)\langle -1,0\rangle & 3 \le t \le 4 \end{cases}$$

The line segment from (x_0, y_0) to (x_1, y_1) can be parameterized as $\langle x_0, y_0\rangle + t\,\langle x_1 - x_0, y_1 - y_0\rangle$ for $0 \le t \le 1$.

Each line segment is of this form but with a shifted range of t

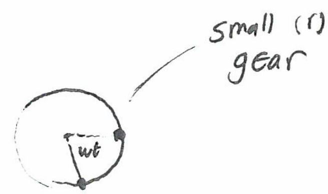
2b)



Assume the small gear rotates at ω rad/sec.

Break motion into two parts:

$$\vec{X}(t) = \underbrace{\vec{X}_{center}(t)}_{\substack{\text{position of}\\ \text{center of}\\ \text{small gear}}} + \underbrace{\vec{X}_{rot}(t)}_{\substack{\text{position of}\\ \text{point}\\ \text{relative}\\ \text{to center}\\ \text{of small}\\ \text{gear.}}}$$



Look at $\vec{X}_{rot}(t)$ first.

In time t, angle swept is ωt.

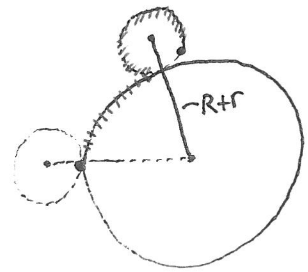
But it is in clockwise direction.

$\theta(t) = 0 - \omega t.$

$\vec{X}_{rot}(t) = r \langle \cos(-\omega t), \sin(-\omega t) \rangle$

Now look at behavior of center of small gear.

$|\vec{X}_{center}(t)| = R+r$ because gears always stay touching



We need angle swept in order to find $\vec{X}_{center}$.

Observe that the arc length swept by smaller gear equals arclength swept along larger gear (by the no slip condition)

Angle swept by small gear: ωt.

Arclength swept along small gear: $r\omega t$.

Angle swept along large gear: $\frac{r\omega t}{R}$ ← contact point is at radius R

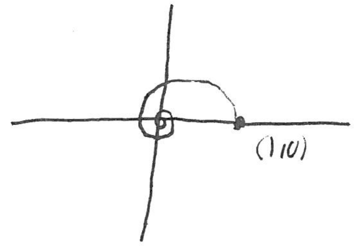
Relative to center of large gear, angle to the center of small gear is

$$\pi - \frac{r\omega t}{R}$$

Hence $\vec{X}_{center}(t) = (R+r)\left\langle \cos\left(\pi - \frac{r\omega t}{R}\right), \sin\left(\pi - \frac{r\omega t}{R}\right)\right\rangle$

So
$$\boxed{\vec{X}(t) = \left\langle r\cos\omega t + (R+r)\cos\left(\pi - \frac{r\omega t}{R}\right), -r\sin\omega t + (R+r)\sin\left(\pi - \frac{r\omega t}{R}\right)\right\rangle \quad \text{for } 0 \le t < \infty}$$

3 a)



radius is $|\vec{X}(t)| = e^{-t}$.
angle is t

b) $\vec{X}(t) = \langle e^{-t}\cos t,\ e^{-t}\sin t \rangle$

$\vec{V}(t) = \frac{d}{dt}\vec{X}(t) = \langle -e^{-t}\cos t - e^{-t}\sin t,\ -e^{-t}\sin t + e^{-t}\cos t \rangle$

$|\vec{V}(t)|^2 = \vec{V}\cdot\vec{V} = (e^{-t}\cos t + e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2$

$= e^{-2t}\left[(\cos t + \sin t)^2 + (\cos t - \sin t)^2\right]$

$= e^{-2t}\left[\cos^2 t + \cancel{2\sin t\cos t} + \sin^2 t + \cos^2 t - \cancel{2\sin t\cos t} + \sin^2 t\right]$

$|\vec{V}(t)|^2 = 2e^{-2t} \implies |\vec{V}(t)| = \sqrt{2}\,e^{-t}$

c) $S = \int_0^\infty |V(t)|\,dt = \int_0^\infty \sqrt{2}\,e^{-t}\,dt = \sqrt{2}$

d) Never reaches origin, as e^{-t} always positive.
Circles at fixed rate. Circles ∞ly many times

e) It's a curve of finite length that circles the origin ∞ly many times! Quite surprising

f) $\vec{X}(t) = \langle \frac{1}{t}\cos t,\ \frac{1}{t}\sin t \rangle$ for $1 \le t < \infty$.
It is like harmonic series, which decays but has convergent sum.
[calculation omitted]

4 a) $U(t,x) = \sin(x-ct)$

$$\partial_t U = -c\cos(x-ct) \qquad \partial_x U = \cos(x-ct)$$
$$\partial_{tt} U = -c^2 \sin(x-ct) \qquad \partial_{xx} U = -\sin(x-ct)$$

We observe $\partial_{tt} U - c^2 \partial_{xx} U = -c^2\sin(x-ct) + c^2 \sin(x-ct)$

$$= 0.$$

For $\sin(x-ct)$ to oscillate once in time T

$$c \cdot T = 2\pi$$
$$T = 2\pi/c \quad \text{is period (sec/cycle)}$$
$$f = \frac{c}{2\pi} \quad \text{is freq (cycle/sec)}$$

$\sin(x-ct)$ is a wave moving to the <u>right</u> with speed <u>c</u>.

If $f(x)$ is any function, $f(x-ct)$ is function translated to right by ct. That is, it moves distance ct in time t. So speed is c.

4b)

$$T(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} = \frac{1}{\sqrt{4\pi}} t^{-1/2} e^{-x^2/4t}$$

$$\partial_t T = \frac{1}{\sqrt{4\pi}} \left(-\frac{1}{2} t^{-3/2} e^{-x^2/4t} + t^{-1/2} e^{-\frac{1}{4}x^2 t^{-1}} \frac{x^2}{4t^2} \right)$$

$$= \frac{1}{\sqrt{4\pi}} \left(-\frac{1}{2} t^{-3/2} e^{-x^2/4t} + \frac{1}{4} \frac{x^2}{t^{5/2}} e^{-x^2/4t} \right)$$

$$\partial_x T = \frac{1}{\sqrt{4\pi}} \left(t^{-1/2} e^{-x^2/4t} \cdot \left(-\frac{2x}{4t} \right) \right)$$

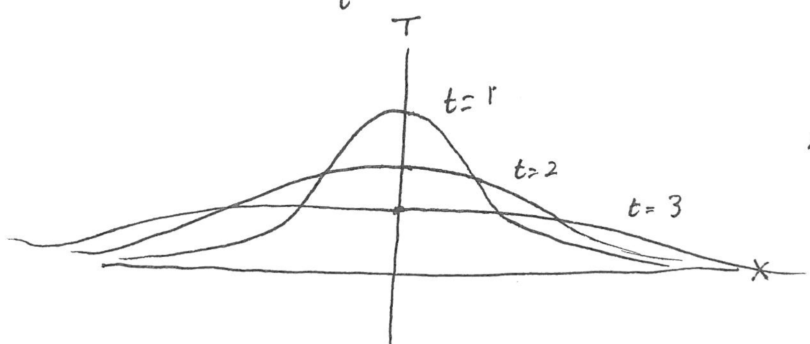
$$= \frac{1}{\sqrt{4\pi}} \left(-\frac{1}{2} t^{-3/2} x e^{-x^2/4t} \right)$$

$$= \frac{1}{\sqrt{4\pi}} \left(-\frac{1}{2} t^{-3/2} \right) x e^{-x^2/4t}$$

$$\partial_{xx} T = \frac{1}{\sqrt{4\pi}} \left(-\frac{1}{2} t^{-3/2} \right) \left(e^{-x^2/4t} + x e^{-x^2/4t} \left(-\frac{2x}{4t} \right) \right)$$

$$= \frac{1}{\sqrt{4\pi}} \left(-\frac{1}{2} t^{-3/2} e^{-x^2/2t} + \frac{1}{4} t^{-5/2} x^2 e^{-x^2/4t} \right)$$

Notice that $\partial_t T = \partial_{xx} T$.



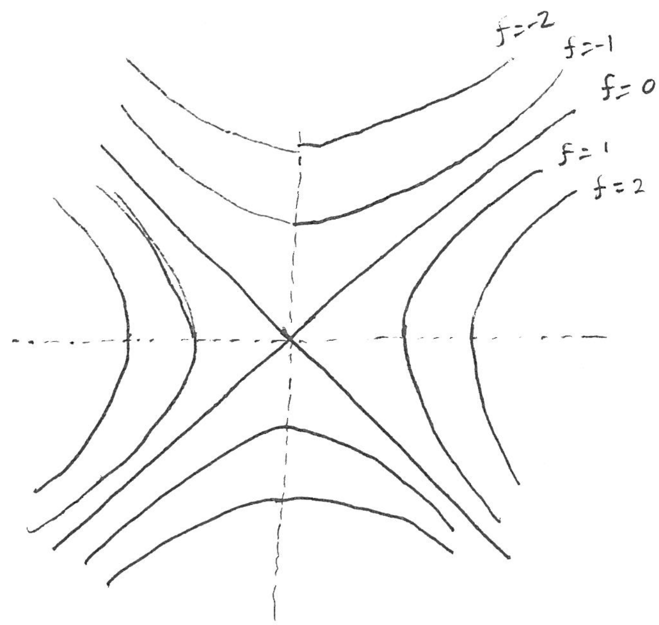
Not to scale

Gaussian that spreads out and decays

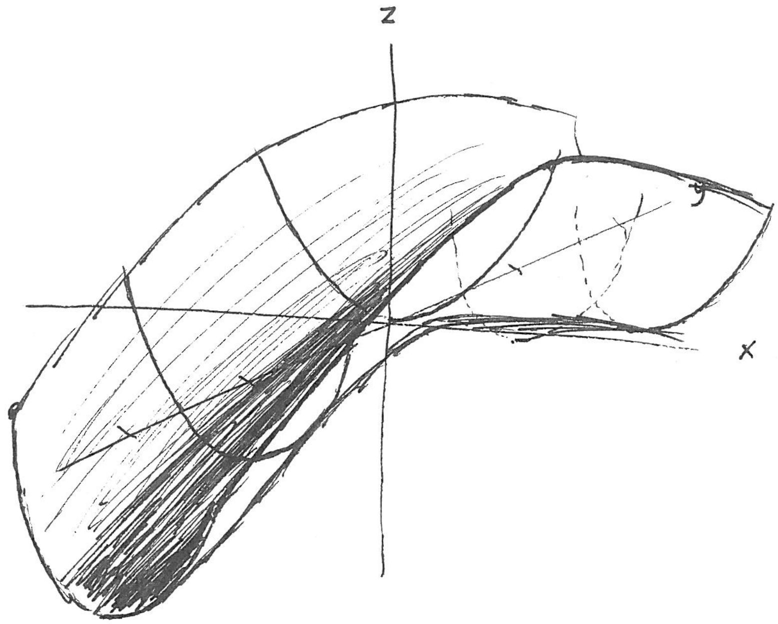
5a) $f(x,y) = x^2 - y^2$

0-level set given by $x^2 - y^2 = 0 \Leftrightarrow x^2 = y^2 \Leftrightarrow |x| = |y|$

c-level set given by $x^2 - y^2 = c$ Hyperbola



5a) $z = x^2 - y^2$



Draw cross sections corresponding to $y = -2, -1, 0, 1, 2$

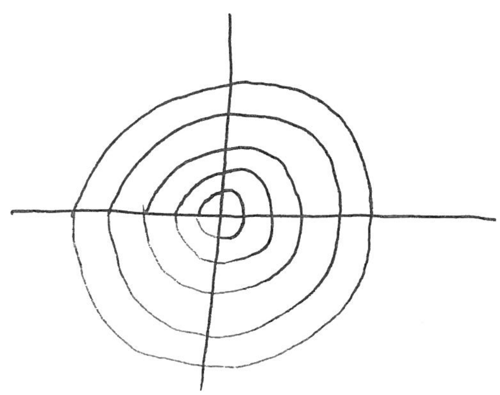
Connect.

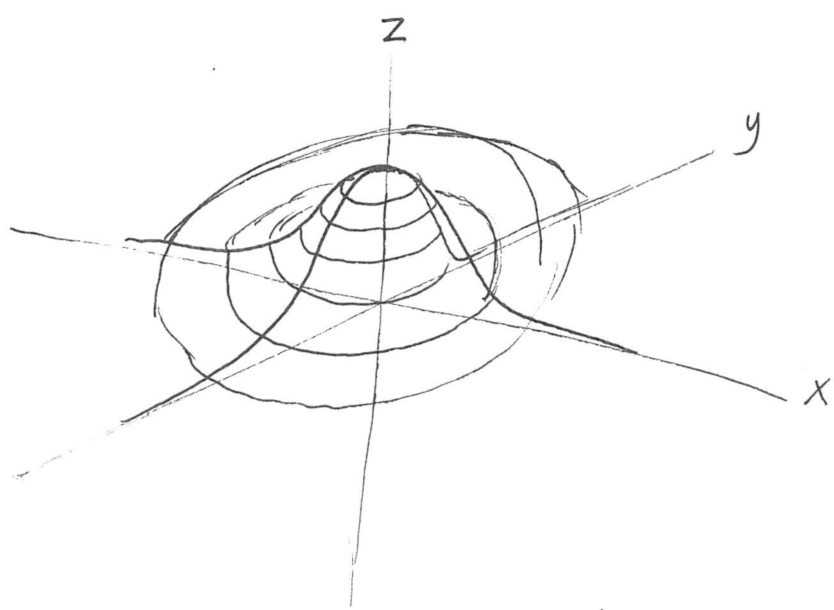
5b) $f(x,y) = e^{-x^2-y^2}$

Level sets: $c = e^{-x^2-y^2}$

$\log c = -x^2-y^2$

So $x^2+y^2 = -\log c$ are circles





Draw $x=0$ & $y=0$ cross-sections.
Connect by level sets

5c)

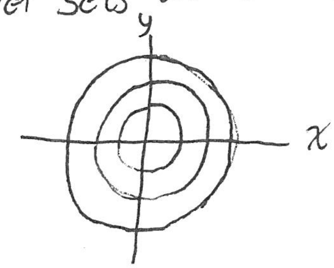
$$f(x,y) = \log \sqrt{x^2+y^2}$$

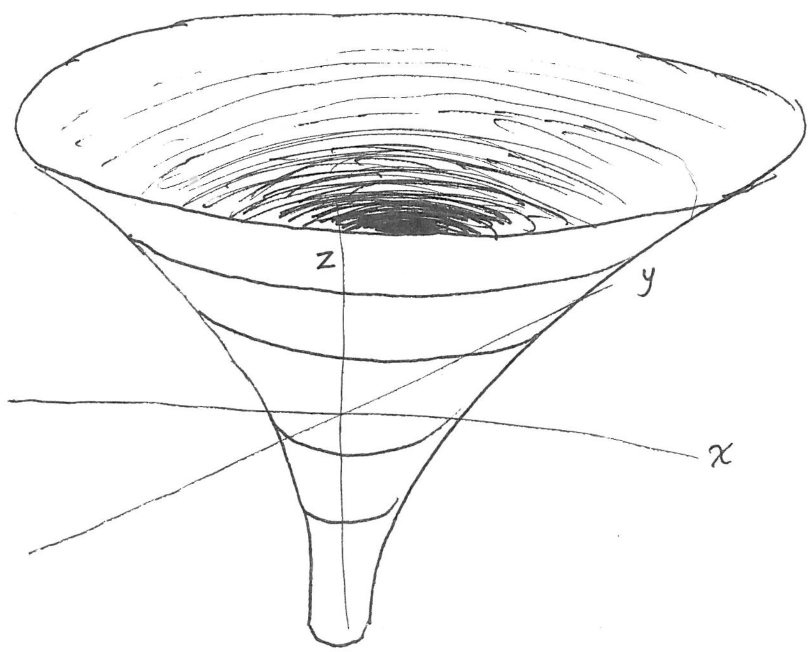
$$c = \log \sqrt{x^2+y^2}$$

$$c = \tfrac{1}{2} \log (x^2+y^2)$$

$$e^{2c} = x^2+y^2$$

Level sets are circles

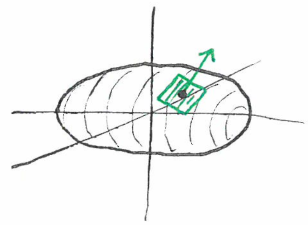




Plot $y=0$ cross section $z = \lg|x|$ and revolve arond z axis (as level sets are circles)

Paul Hand

6 a) $\quad x^2 + 2y^2 + 3z^2 = 9$ is ellipsoid



This surface is the 9-level set of

$$f(x,y,z) = x^2 + 2y^2 + 3z^2$$

Recall that ∇f is perpendicular to level sets of f

$$\vec{\nabla} f(x,y,z) = \langle 2x, 4y, 6z \rangle$$

$$\vec{\nabla} f(2,-1,1) = \langle 4,-4,6 \rangle$$

Hence a normal vector to surface is $\vec{n} = \langle 4,-4,6 \rangle$

The tangent plane is thus

$$\langle 4,-4,6 \rangle \cdot \vec{x} = \langle 4,-4,6 \rangle \cdot \langle 2,-1,1 \rangle$$

$$\langle 4,-4,6 \rangle \cdot \vec{x} = 18$$

$$\boxed{\langle 2,-2,3 \rangle \cdot \vec{x} = 9}$$

6b)

Let $\vec{X} = (x, y, z)$

Plane from 1a is $2x - 2y + 3z = 9$

If we plug in $x = 2.01$ & $y = -0.99$,

$$3z = 9 - 2(2.01) + 2(-0.99)$$

$$\boxed{z = 1}$$

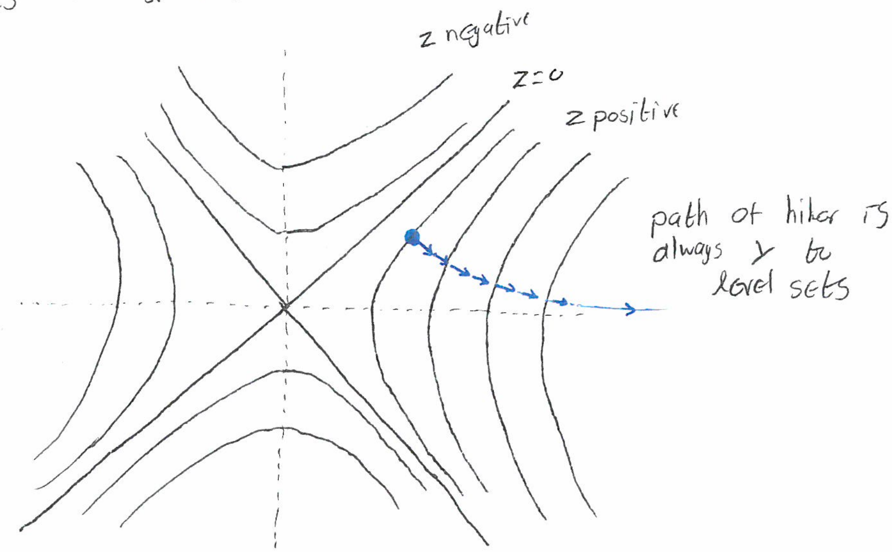
Exact value given by $z = \sqrt{\dfrac{9 - x^2 - 2y^2}{3}}$

$$= 0.99995\ldots$$

Error $\approx 5 \cdot 10^{-5}$

Level sets are hyperbolas $x^2 - y^2 = C$

7a)



b) First, compute 2d direction of steepest ascent,

$$\vec{\nabla} z = \langle 2x, -2y \rangle$$

$$\vec{\nabla} z(2,1) = \langle 4, -2 \rangle$$

Direction of steepest ascent is $\vec{U} = \dfrac{\langle 4,-2\rangle}{|\langle 4,-2\rangle|} = \dfrac{\langle 4,-2\rangle}{\sqrt{16+4}} = \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$

Directional derivative in dir of U is

$$D_{\vec{U}} z = \vec{\nabla} z \cdot \vec{U} = \langle 4,-2\rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$$

$$= 2\sqrt{5}$$

For every unit length travelled in direction $\vec{U}$, there is a rise in z of $2\sqrt{5}$. So, tangent vector in 3d is

$$\boxed{\left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 2\sqrt{5} \right\rangle}$$