Lecture 9

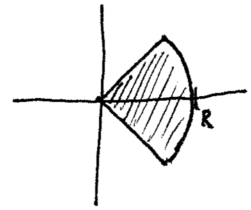
22 july 2014

Double Integrals in Polar

Triple Integrals in cartesian

Warmup:

a)

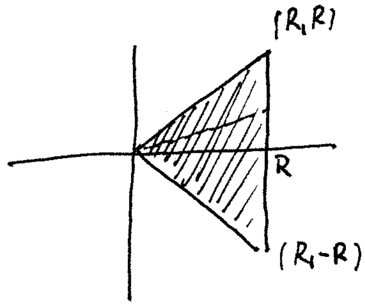


$0 \le r \le R$

$-\pi/4 \le \theta \le \pi/4$

Describe using polar coordinates

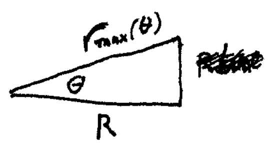
b



$-\pi/4 \le \theta \le \pi/4$

$0 \le r \le R/\cos\theta$

Describe in terms of polar coordinates

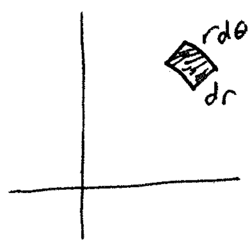


$\cos\theta = R/r_{max}$

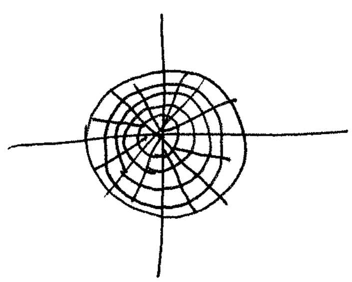
$r_{max}(\theta) = R/\cos\theta$

Area element in Polar

A region at (r, θ) of size dr & $d\theta$ has area $dA = r\,dr\,d\theta$

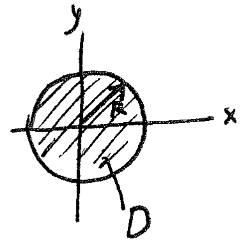


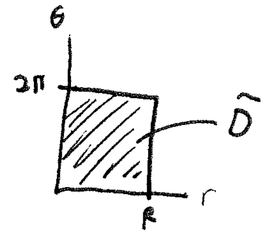
So $\iint_R f\,dA = \iint_R f(r,\theta)\,r\,dr\,d\theta$



like adding up f over all of these "boxes"

Example: Area of circle of radius R





$$\iint_D 1 \, dx\,dy = \iint_{\tilde{D}} 1 \; r\,dr\,d\theta$$

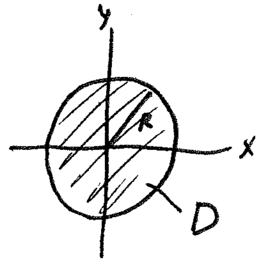
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{R} 1 \; r\,dr \; d\theta$$

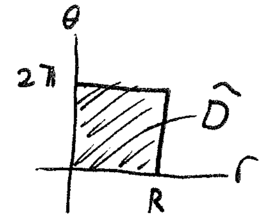
— can separate b/c r integral constant w.r.t. θ

$$= \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{R} r\,dr$$

$$= 2\pi \; \tfrac{1}{2}R^2 = \pi r^2$$

Example: Moment of inertia of disk of radius R, area-mass density ρ (constant)





$$\iint_D (x^2+y^2)\,\rho\,dx\,dy = \iint_{\tilde{D}} r^2 \rho \; r\,dr\,d\theta$$

$$= \rho \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r^3\,dr\,d\theta$$

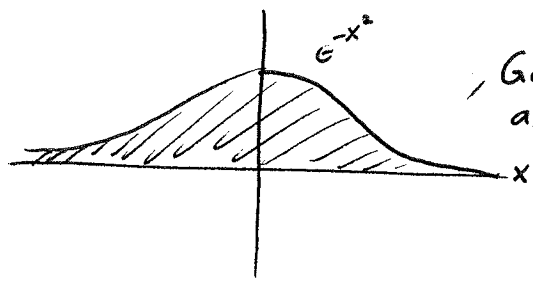
$$= \rho \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{R} r^3\,dr$$

$$= \rho\, 2\pi\, \frac{1}{4} R^4 = \rho \frac{\pi}{2} R^4 = \frac{1}{2}(\pi R^2 \rho) R^2$$

$$= \frac{1}{2} M R^2$$

IMPORTANT Example

Show $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$



Gaussian
aka Bell curve

Most important
probability distribution

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

cant solve by substitution or guessing antiderivative

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

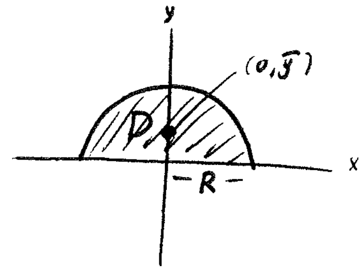
$$= \int_{0}^{2\pi} d\theta \int_{0}^{\infty} e^{-r^2}\,r\,dr$$

$$= 2\pi \left.\frac{e^{-r^2}}{-2}\right|_{0}^{\infty}$$

$$= \pi$$

So $\boxed{I = \sqrt{\pi}}$!!

Example: Find center of mass of half of disk of constant density



$\bar{X} = 0$ by symmetry

$$\bar{y} = \frac{\iint_D y \, \rho \, dx\,dy}{\iint_D \rho \, dx\,dy}$$

Evaluate: $\iint_D \rho \, dx\,dy = \rho \iint_D dx\,dy = \rho \cdot \text{Area of } D$

$$= \rho \tfrac{1}{2} \pi R^2$$

$$\iint_D y \, \rho \, dx\,dy = \int_{\theta=0}^{\pi}\int_{r=0}^{R} r\sin\theta \, \rho \, r\,dr\,d\theta$$

$$= \rho \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{r=0}^{R} r^2\,dr$$

$$= \rho \left[-\cos\pi\right|_0^{\pi} \cdot \tfrac{1}{3} R^3$$

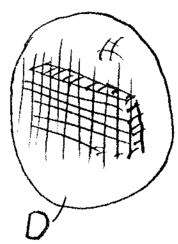
$$= \rho \; 2 \; \tfrac{1}{3} R^3$$

So $\bar{y} = \dfrac{\frac{2}{3}\cancel{\rho} R^3}{\cancel{\rho}\,\frac{1}{2}\cancel{\pi}R^2} = \dfrac{4}{3\pi} R = \dfrac{4R}{3\pi}$

Triple Integrals

Let D be a 3d region.

$$\iiint_D f(x,y,z)\,dV$$ is the volume weighted sum of f.



Break into small little cubes $\Delta x \times \Delta y \times \Delta z$

$$\iiint_D f\,dV \approx \sum f(x_i, y_i, z_i)\,\Delta x\,\Delta y\,\Delta z$$

Riemann sum

In cartesian, $dV = dx\,dy\,dz$

To evaluate, express region as range of x, a possibly x-dependent range of y, and a possibly x,y-dependent range of z.

(or in any other order of x,y,z)

Average value of $f(x,y,z) = xyz$ over $[0,1]\times[0,1]\times[0,1]$

$$\bar{f} = \frac{\iiint_D f\,dV}{\iiint_D dV} = \;?$$

$$\iiint_D dV = \operatorname{Vol}(D) = 1\cdot1\cdot1 = 1$$

$$\iiint_D f\,dV = \iiint_D xyz\,dx\,dy\,dz$$

$$= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} xyz\,dz\,dy\,dx$$

$$= \int_{x=0}^{1} x\,dy \int_{y=0}^{1} y\,dy \int_{z=0}^{1} z\,dz$$

$$= \tfrac{1}{2}x^2\Big|_0^1 \;\tfrac{1}{2}y^2\Big|_0^1 \;\tfrac{1}{2}z^2\Big|_0^1$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{8}.$$

So $\bar{f} = \frac{1}{8}$

Activity: Cube $[0,1] \times [0,1] \times [0,1] = D$

Compute $\bar{x}$ of center of mass.

Assume $\rho \equiv$ constant

$$\bar{x} = \frac{\iiint_D x \rho \, dV}{\iiint_D \rho \, dV} = \frac{\cancel{\rho} \iiint_D x \, dV}{\cancel{\rho} \iiint_D dV} = \frac{\iiint x \, dV}{1}$$

Vol(cube) = 1

$$\iiint_D x \, dV = \int_0^1 \int_0^1 \int_0^1 x \, dx \, dy \, dz$$

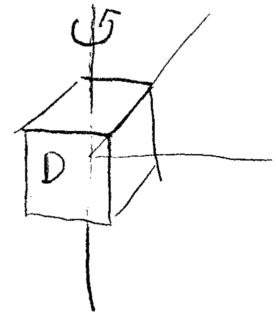
$$= \int_0^1 x \, dx \int_0^1 dy \int_0^1 dz$$

$$= \frac{1}{2} \cdot 1 \cdot 1$$

$$= 1/2$$

Example: Find moment of inertia of a cube of width L about the axis shown. Constant density ρ.

Align cube w/ coordinate axes.



$$I = \iiint_D d^2(x,y,z)\,\rho\, dV$$

If rotation axis is z-axis $\quad d(x,y,z) = \sqrt{x^2+y^2}$

$$I = \iiint_D (x^2+y^2)\,\rho\; dx\,dy\,dz$$

Specify D in cartesian coordinates

$$-\tfrac{L}{2} \leq x \leq \tfrac{L}{2}$$
$$-L/2 \leq y \leq L/2$$
$$-L/2 \leq z \leq L/2$$

$$I = \int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \rho\,(x^2+y^2)\,dx\,dy\,dz$$

$$= \rho\left[\iiint x^2\,dx\,dy\,dz + \iiint y^2\,dx\,dy\,dz\right]$$

$$= \rho\left[\int_{-L/2}^{L/2} x^2 dx \int_{-L/2}^{L/2} dy \int_{-L/2}^{L/2} dz + \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} y^2 dy \int_{-L/2}^{L/2} dz\right]$$

$$= \rho\left[\tfrac{1}{3}x^3\Big|_{-L/2}^{L/2}\, L\, L + L\left(\tfrac{1}{3}y^3\Big|_{-L/2}^{L/2}\right) L\right]$$

$$= \rho\left[\tfrac{2}{3}L^5/8 + \tfrac{2}{3}L^5/8\right] = \rho\,\frac{4}{3\cdot 8}L^5 = \frac{1}{6}(\rho L^3)L^2$$

$$= \frac{1}{6}ML^2$$