Lecture 6 - Constrained optimization and Lagrange multipliers - 7/14/2014 — Interphase 2014 Calc 3

19. Critical points and the second derivative test

    a. If $\vec{\nabla} f(x, y) = 0$, then (x, y) is a critical point of $f(x, y)$.

    b. A critical point can be a local max, a local min, or a saddle point.

    - A local max curves down in all directions
    - A local min curves up in all direction
    - A saddle point curves down in some directions and up in other directions

    c. *Second derivative test in 2d:* Suppose (x_0, y_0) is a critical point for f.

    $$\text{Let } H = \begin{pmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix}.$$

    - If $\det(H) > 0$, then (x_0, y_0) is either a local max or local min
    - If $\det(H) < 0$, then (x_0, y_0) is a saddle point
    - If $\det(H) = 0$, then the second derivative test is inconclusive.

20. Optimization problems

    a. An optimization problem is of the form

    $$\min f(x, y, z) \text{ subject to } g(x, y, z) = 0.$$

    The function f is the *objective*. The equation $g = 0$ is a *constraint*.

    b. To pose an optimization problem:

    - Quantify the search space. Determine the relevant variables.
    - Quantify the objective.
    - Simplify the objective (by removing square roots, logs, etc as appropriate)
    - Simplify the search space by using the constraint to remove one of the variables

    c. To solve an optimization problem without constraints:

    - Find the critical points
    - If needed, use the second derivative test to determine if each point is a min/max/saddle.

    d. To use Lagrange multipliers to solve the problem

    $$\min f(x, y, z) \text{ subject to } g(x, y, z) = 0,$$

    1. Form the augmented function

       $$L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$$

    2. Set all partial derivatives of L equal to zero
    3. Solve for x, y, z.