Lecture 4

9 july 2014

Functions of several variables

Partial derivatives

15) Functions of Several Variables

$$f(x,y) \quad \text{or} \quad f(x,y,z) \quad \text{or} \quad f(\vec{x})$$

Takes (x,y) or (x,y,z) and returns a number.

Example:

$f(x,t) = \sin(x - ct)$ sound wave

$$f(x,y,z,t) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{x^2+y^2+z^2}{4t}}$$

time-dependent concentration of diffusing chemical in 3d

How can we visualize these functions?

15b)

The level set with value c of the function $f(x,y)$ is the set of (x,y) such that $f(x,y) = c$.

- 2d : level curve
- 3d : level surface
- generally : level set

Visualize a function by drawing several level sets with different values of c.

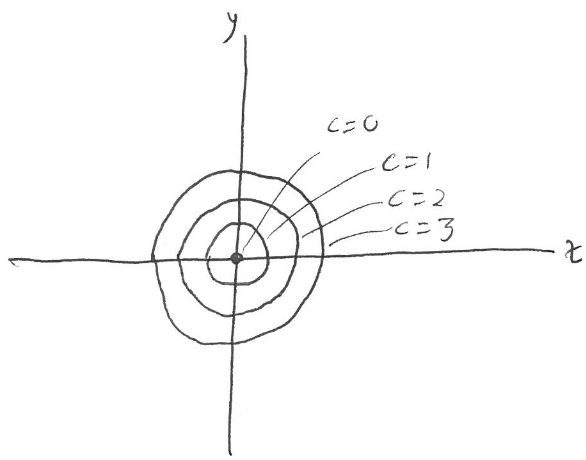
Like topographic map.

Example: Draw several level sets of $f(x,y) = x^2+y^2$.

Level set of value 0: $x^2+y^2=0$. Just the point $(0,0)$

Level sets of negative value: Impossible

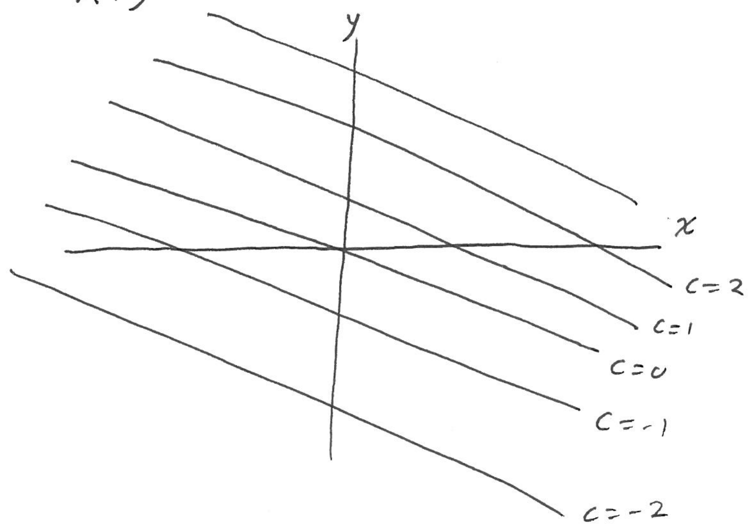
Level set of value $c>0$: $x^2+y^2=c$ Circle w/ radius $\sqrt{c}$



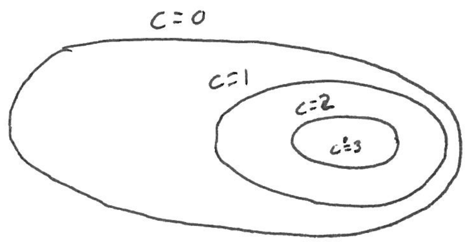
Activity:

Draw the level sets of $f(x,y) = x+2y$

$x+2y=c$ is line w/ normal vector $\langle 1,2 \rangle$



Activity:

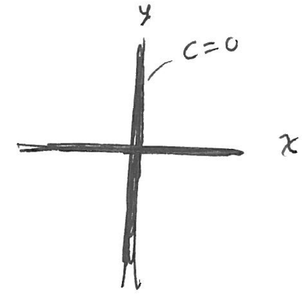


Here are level sets of $f(x,y)$.

Where is f the steepest?

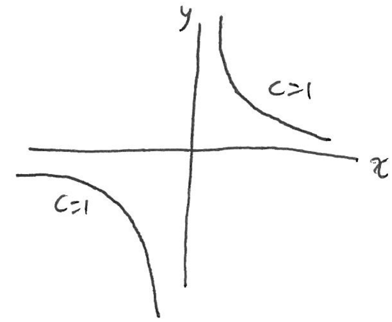
Where is f the flattest?

Example: Find level sets of $f(x,y)=xy$

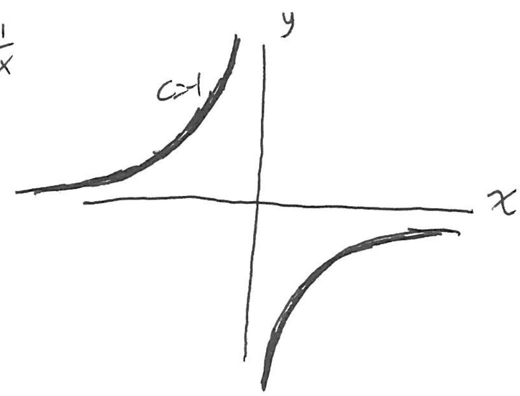


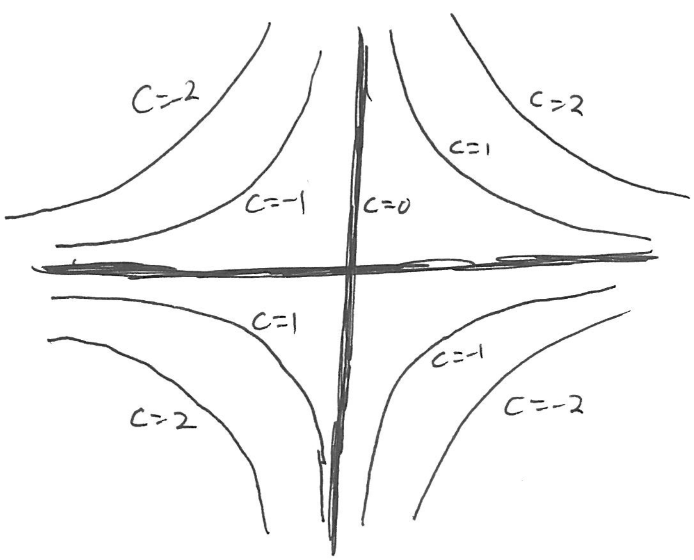
level set w/ value 0: $xy=0 \Rightarrow x=0$ or $y=0$

level set w/ value 1: $xy=1 \Rightarrow y=\frac{1}{x}$



level set w/ value -1: $xy=-1 \Rightarrow y=\frac{-1}{x}$





Activity:

a) Draw the level set w/ value 1 of

$$f(x,y,z) = x^2+y^2+z^2$$

b) Draw the level set w/ value 1 of

$$\cancel{g(x,y) = x^2+y^2+z}$$

$$\cancel{x^2 + \frac{y^2}{4}}$$

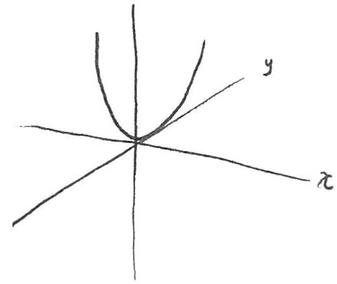
$$g(x,y) = x^2+\frac{y^3}{4}$$

15c) Visualizing by drawing 3d surface

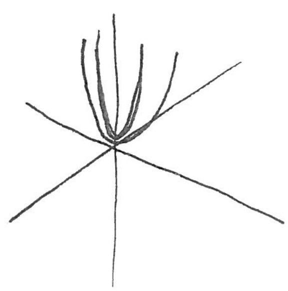
To Plot surface $Z = f(x, y)$:

- Plot cross sections for values of x
- Plot cross sections for values of y
- Connect. Pay attention to level sets

Ex: plot $Z = x^2 + y^2$

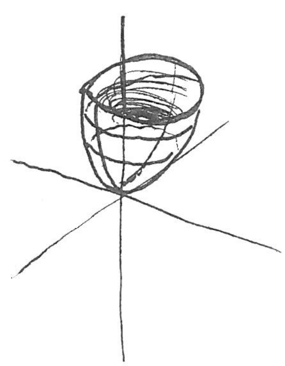


$y=0$ cross section
$Z = x^2$



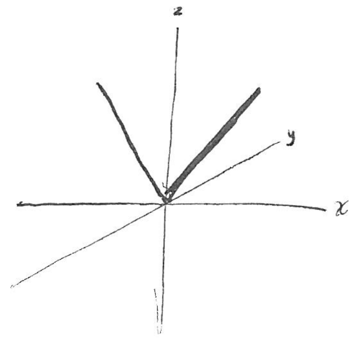
$x=0$ cross section
$Z = y^2$

Note: level sets of x^2+y^2 are circles.

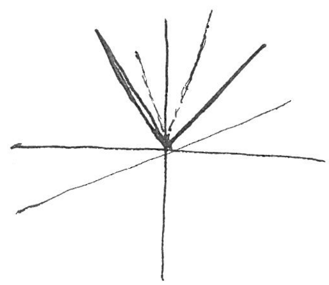


Ex: Plot $z = |x| + |y|$

Plot $y=0$ cross section



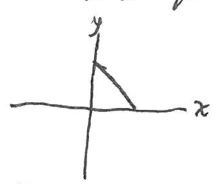
Plot $x=0$ cross section



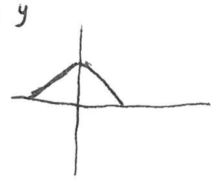
Note: level sets of $|x|+|y|$ are diamonds

why: $|x|+|y| = c$. This shape is mirror symmetric about $x=0$, & about $y=0$; plot in first quadrant:

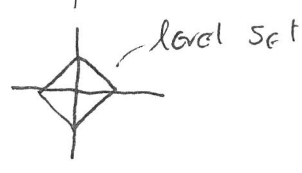
If $x \geq 0, y \geq 0$ $x+y=c$



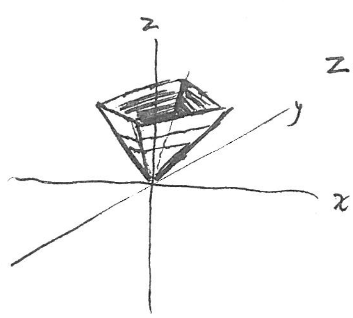
Enforce symmetry wrt y



Enforce sym wrt x



Connect 3D sketch

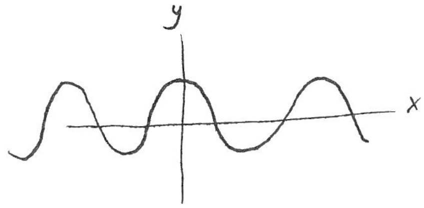


$z = |x| + |y|$

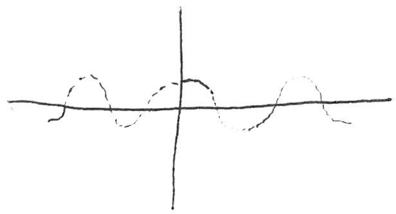
15d Visualizing by sequence of sketches

Visualize $y = e^{-t} \cos x$

At $t=0$

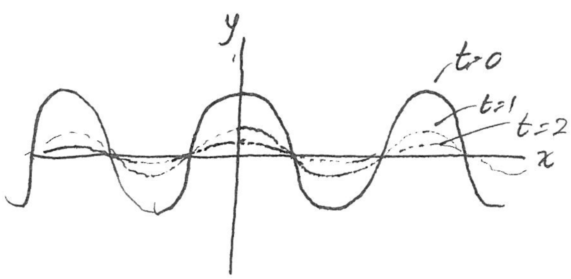


At $t=1$

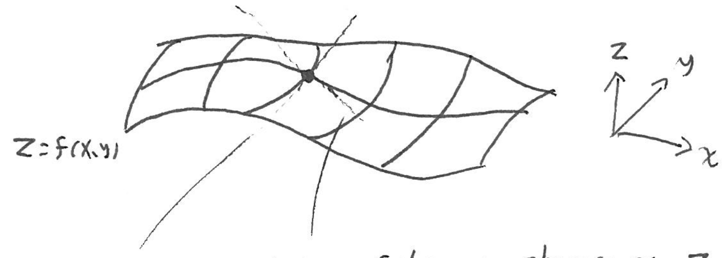


And so on

Combine



16 e)



$\partial_y f$ =
rate of change
of z wrt y
=
slope in y direction

$\partial_x f$ = rate of change of z = slope in x direction
wrt x

Example: $f(x,y) = x^2 + 2xy + y^2$

$f_x(x,y) = 2x + 2y$ $\qquad f_y(x,y) = 2x + 2y$

$f_{xx}(x,y) = 2$ $\qquad f_{yx}(x,y) = 2$

$f_{xxx}(x,y) = 0$ $\qquad f_{yy}(x,y) = 2$

Example: $f(x,y) = \log(x + e^y)$. ~~Assume $x > 0$. Compute~~

$\partial_x f(x,y) = \dfrac{1}{|x + e^y|}$ $\qquad \partial_y f(x,y) = \dfrac{e^y}{|x + e^y|}$

Example: Show that $\phi(x,y) = \log(x^2+y^2)$ satisfies $\partial_{xx}\phi(x,y) + \partial_{yy}\phi(x,y) = 0$

Compute $\partial_x\phi$, $\partial_y\phi$, $\partial_{xx}\phi$, $\partial_{yy}\phi$ and show equality satisfied.

$\phi(x,y) = \log(x^2+y^2)$

$\partial_x\phi(x,y) = \dfrac{2x}{x^2+y^2}$ $\qquad \partial_y\phi = \dfrac{2y}{x^2+y^2}$

$\partial_{xx}\phi(x,y) = \dfrac{2}{x^2+y^2} + \dfrac{2x(-1)2x}{(x^2+y^2)^2}$ $\qquad \partial_{yy}\phi(x,y) = \dfrac{2}{x^2+y^2} - \dfrac{4y^2}{(x^2+y^2)^2}$

$= \dfrac{2}{x^2+y^2} - \dfrac{4x^2}{(x^2+y^2)^2}$

So $\partial_{xx}\phi(x,y) + \partial_{yy}\phi(x,y) = \dfrac{2}{x^2+y^2} - \dfrac{4x^2}{(x^2+y^2)^2} + \dfrac{2}{x^2+y^2} - \dfrac{4y^2}{(x^2+y^2)^2}$

$= \dfrac{4}{x^2+y^2} - 4\dfrac{(x^2+y^2)}{(x^2+y^2)^2} = 0$ ✓

16f) Linear Approximations of Functions

How much does $f(x,y)$ change if (x,y) perturbed by $(\Delta x, \Delta y)$?

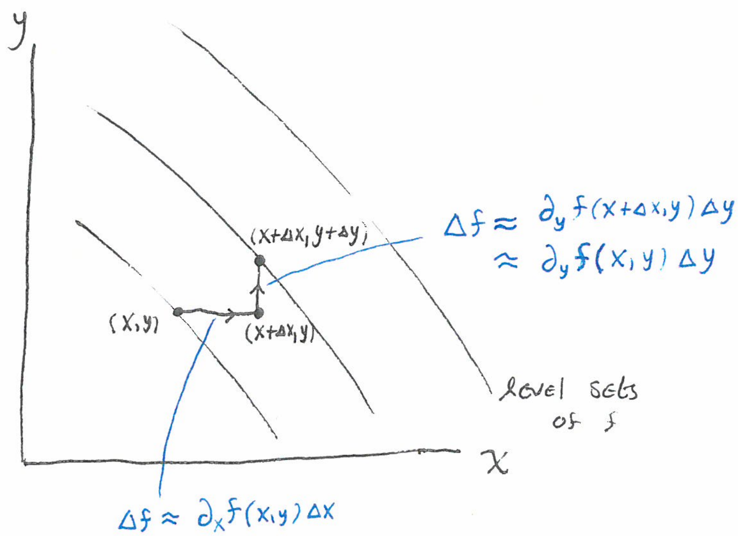
$$\underbrace{f(x+\Delta x, y+\Delta y) - f(x,y)}_{\Delta f} \approx \underbrace{\partial_x f(x,y)\,\Delta x}_{\text{change due to stepping in } x} + \underbrace{\partial_y f(x,y)\,\Delta y}_{\text{change due to stepping in } y}$$

Comments:

- Like 1d Taylor series

  $$f(x+\Delta x) - f(x) \approx f'(x)\,\Delta x$$

- Visually,



Total $\Delta f \approx \partial_x f(x,y)\Delta x + \partial_y f(x,y)\Delta y$

- In terms of tangent plane to $z = f(x,y)$:

  Change in f approximated by change in height of ~~(approximating)~~ tangent plane.

  $$(-\partial_x f, -\partial_y f, 1)\cdot(x,y,z) = b \quad \text{is tangent plane}$$

  $$(-\partial_x f, -\partial_y f, 1)\cdot(x+\Delta x, y+\Delta y, z+\Delta z) = b$$

  So

  $$(-\partial_x f, -\partial_y f, 1)\cdot(\Delta x, \Delta y, \Delta z) = 0$$

  So $\Delta z = \partial_x f\,\Delta x + \partial_y f\,\Delta y$