30 jun 2014

Calculus 3 - Interphase 2014

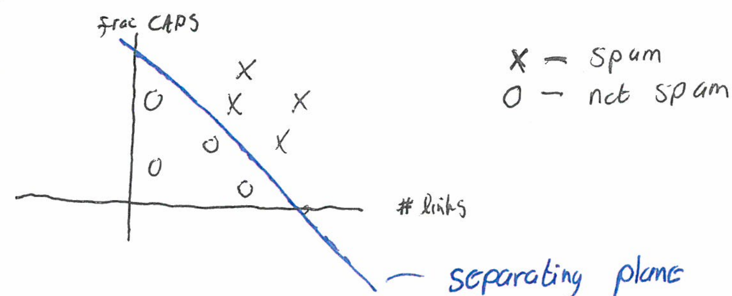
Multivariable Calculus:

- vectors
- differentiating functions of multiple variables
- integrating over curves and surfaces

Problems we'll solve:

(1) Spam detection

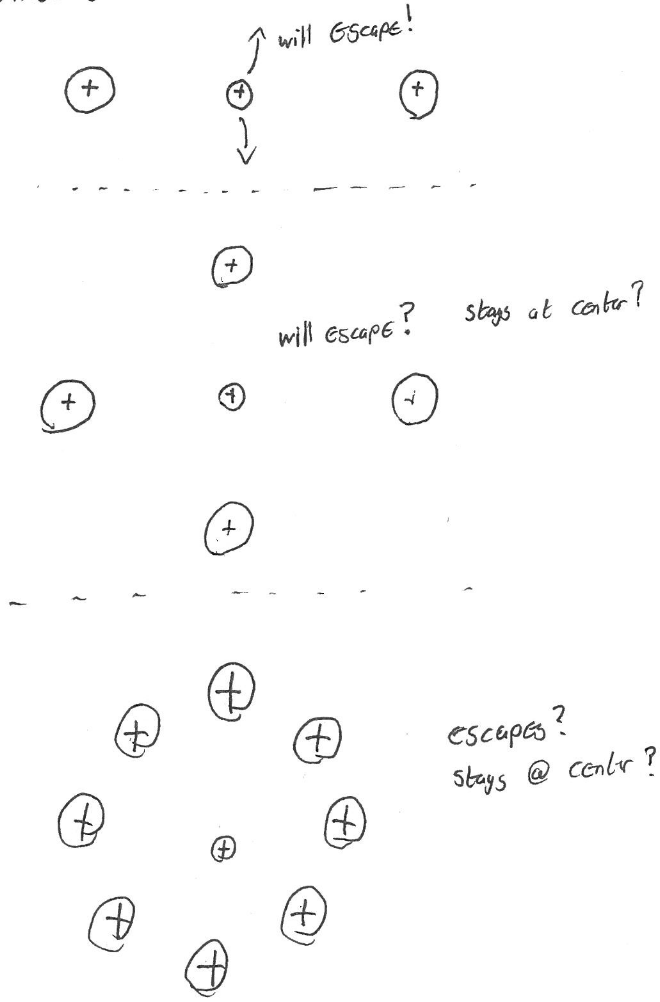
- Have human classify many emails spam/not spam
- Form feature vector for all email (# links, fraction CAPS, # special chars.
- Find plane separating spam & not-spam
- Classify new emails relative to this plane



Tools:
- vectors
- dot products
- planes

(2) Particle Trapping

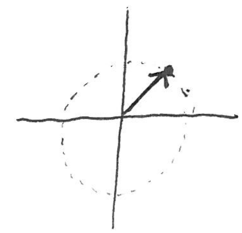
Using electrostatic repulsion ONLY, can you keep a charged particle fixed?



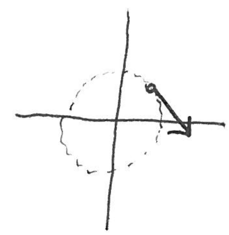
Tools:
- Partial Derivatives
- Double Integrals
- Line integrals
- Green's Theorem

1a) Examples of Vectors

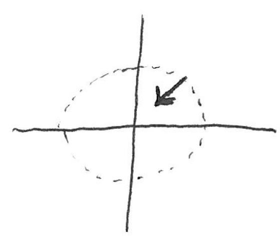
Object twirled by rope has position



and velocity



and is subject to a force

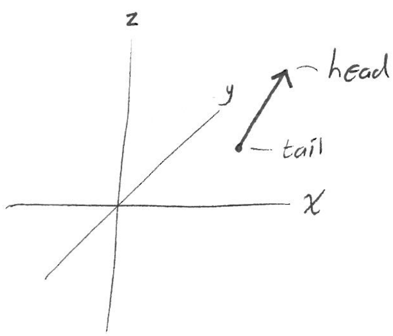


Can be 3d or higher

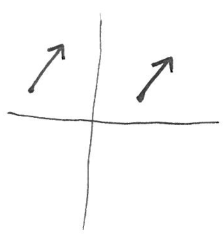
Eg 1000 x 1000 grayscale image is a vector w/ 1,000,000 dimensions

What is a vector?

- Quantity with magnitude + direction
- Collection of multiple numbers
- arrow with a head and tail



The location of the tail doesnt matter. All that matters is position of head relative to tail.

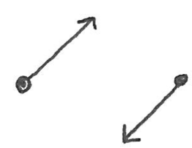


these are equal as vectors

1b)

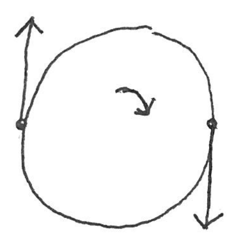
Velocity is a vector — magnitude + direction

Speed is a scalar — magnitude, no direction



- different velocities
- same speeds

Merry go round



These points have
- different velocities
- same speed.

Vector Notation

$\vec{X}$ or $\mathbf{X}$ or X is a vector

bold

In 3D

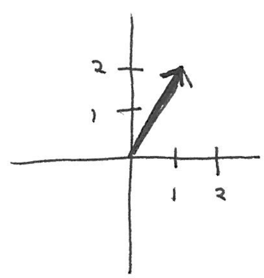
$$\vec{X} = \langle a, b, c \rangle = (a, b, c) = a\hat{i} + b\hat{j} + c\hat{k}$$

first component second component third component

In 2D

$$\vec{X} = \langle a, b \rangle = (a, b) = a\hat{i} + b\hat{j}$$

Draw $\langle 1, 2 \rangle$



Note: May also write

$$\vec{X} = \langle X_1, X_2, X_3 \rangle$$

or

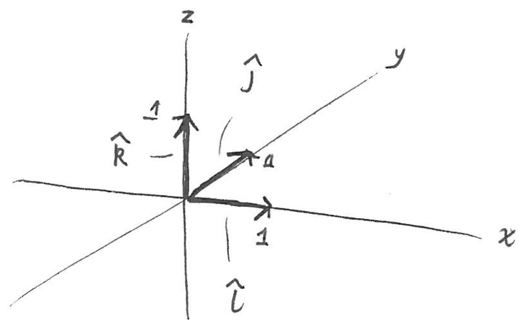
$$\vec{F} = \langle F_x, F_y, F_z \rangle$$

Activity:

Draw $\langle -3, 1 \rangle$

Draw $-2\hat{i} - 3\hat{j} + \hat{k}$

1c) Coordinate Vectors



$\hat{i} = \langle 1,0,0 \rangle$ in 3d or $\langle 1,0 \rangle$ in 2d

$\hat{j} = \langle 0,1,0 \rangle$ in 3d or $\langle 0,1 \rangle$ in 2d

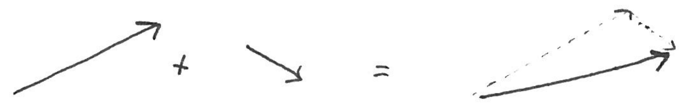
$\hat{k} = \langle 0,0,1 \rangle$ in 3d

The hat "∧" means they have length 1.

2a) $\langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle = \langle x_1 + y_1, x_2 + y_2 \rangle$

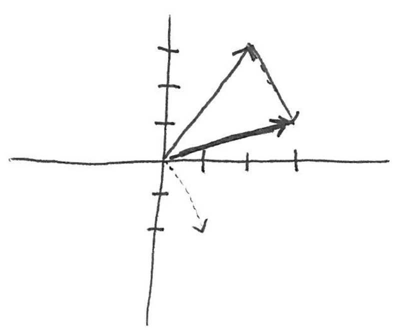
$\langle x_1, x_2, x_3 \rangle + \langle y_1, y_2, y_3 \rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3 \rangle$

Example: $\langle 2, 3 \rangle + \langle -2, -2 \rangle = \langle 0, 1 \rangle$

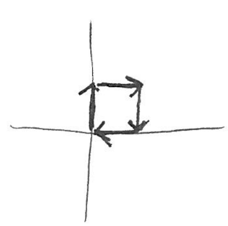
2b) 

Does it agree with 2a? Yes

Example: $\langle 2, 3 \rangle + \langle 1, -2 \rangle = \langle 3, 1 \rangle$



Example: What is sum of $\uparrow + \rightarrow + \downarrow + \leftarrow$? 0

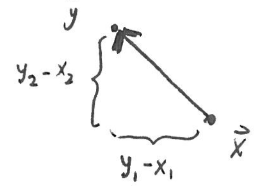


you get back to where you start

3) Vector subbraction

$$\langle x_1, x_2\rangle - \langle y_1, y_2\rangle = \langle x_1 - y_1, x_2 - y_2\rangle$$

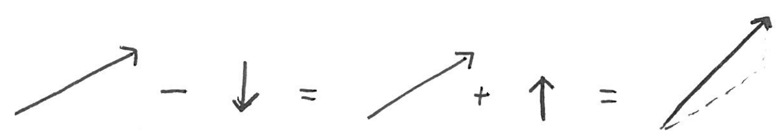
The vector from $\vec{x}$ to $\vec{y}$ is $\vec{y} - \vec{x}$



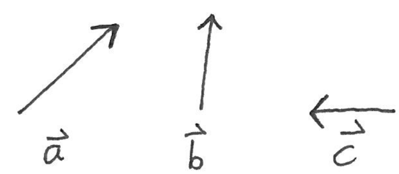
How will you remember if it is $\vec{y} - \vec{x}$ or $\vec{x} - \vec{y}$?

- The vector from $\vec{0}$ to $\vec{y}$ is $\vec{y} - \vec{0} = y$ not $\vec{0} - \vec{y} = -\vec{y}$.
- It is what needs to be added to $\vec{x}$ in order to get $\vec{y}$.

Subtracting a vector is same as adding its negative



Activity: Using addition and subtraction, combine these vectors to get 0.



For scalar c, and vector $\vec{x}$

5c) $|c\vec{x}| = |c||\vec{x}|$

absolute

Proof:

Let $\vec{x} = (x_1, x_2, x_3)$

$$c\vec{x} = (cx_1, cx_2, cx_3)$$

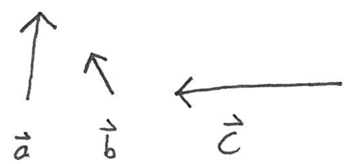
$$|c\vec{x}| = \sqrt{(cx_1)^2 + (cx_2)^2 + (cx_3)^2}$$

$$= \sqrt{c^2(x_1^2 + x_2^2 + x_3^2)}$$

$$= |c| \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$= |c| \, |\vec{x}|.$$

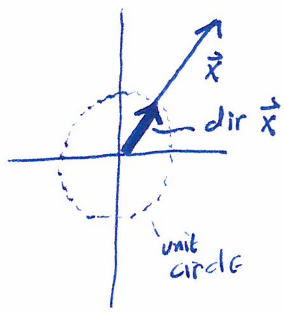
Activity: Using addition, subtraction, scalar multiplication, combine these vectors to form 0.



Activity

For What value of c is $c\langle 1,1,1\rangle$ a unit vector?

6a) Direction of a vector



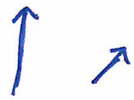
$$\operatorname{dir} \vec{x} = \frac{\vec{x}}{|\vec{x}|}$$

What is length of dir $\vec{x}$?

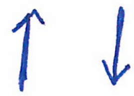
$$\left| \frac{\vec{x}}{|\vec{x}|} \right| = \frac{1}{|\vec{x}|} |\vec{x}| = \underline{1}.$$

6b) $\vec{x} = |\vec{x}| \operatorname{dir} \vec{x}$

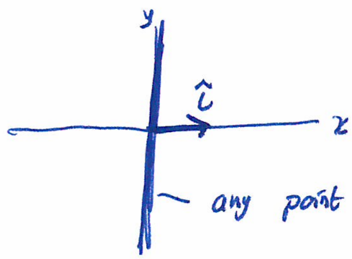
If two vectors have equal length & direction, they are equal

 angle is $\pi/4$

7a) Example:  angle is $\frac{3\pi}{4}$ 

 angle is π

Example: Draw all points in 2D with angle $\pi/2$ from $\hat{i}$



any point on this line has angle $\frac{\pi}{2}$ from $\hat{i}$

Activity

In 2d, plot all points w/ angle $\pi/4$ or less from $\hat{i}+\hat{j}$

In 3d, plot all points w/ angle $\pi/6$ from $\hat{i}$.

8b) $\vec{X} \cdot \vec{X} = |\vec{X}|^2$

Proof in 3d

Let $\vec{X} = (X_1, X_2, X_3)$

$\vec{X} \cdot \vec{X} = X_1^2 + X_2^2 + X_3 = |\vec{X}|^2$.

8c) Prove $(\vec{U} + \vec{V}) \cdot \vec{X} = \vec{U} \cdot \vec{X} + \vec{V} \cdot \vec{X}$

Let $\vec{U} = (U_1, U_2, U_3)$

$\vec{V} = (V_1, V_2, V_3)$

$\vec{X} = (X_1, X_2, X_3)$

$\vec{U} + \vec{V} = (U_1 + V_1, U_2 + V_2, U_3 + V_3)$

$(\vec{U} + \vec{V}) \cdot \vec{X} = X_1(U_1 + V_1) + X_2(U_2 + V_2) + X_3(U_3 + V_3)$

$= X_1 U_1 + X_2 U_2 + X_3 U_3$

$+ X_1 V_1 + X_2 V_2 + X_3 V_3$

$= \vec{X} \cdot \vec{U} + \vec{X} \cdot \vec{V}$.

8d) Dot product angle formula

$$\vec{x} \cdot \vec{y} = |\vec{x}|\,|\vec{y}| \cos\theta \quad \text{for any } \vec{x} \text{ and } \vec{y}.$$

where θ is angle between $\vec{x}$ and $\vec{y}$

Example: Find angle between $\underbrace{\langle 1, 2 \rangle}_{\vec{x}}$ and $\underbrace{\langle -2, 1 \rangle}_{\vec{y}}$

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}|\,|\vec{y}|}$$

$$\vec{x} \cdot \vec{y} = 1(-2) + 2(1) = 0.$$

$$|\vec{x}| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$|\vec{y}| = \sqrt{5}$$

$$\cos\theta = 0 \implies \boxed{\theta = \pi/2}$$

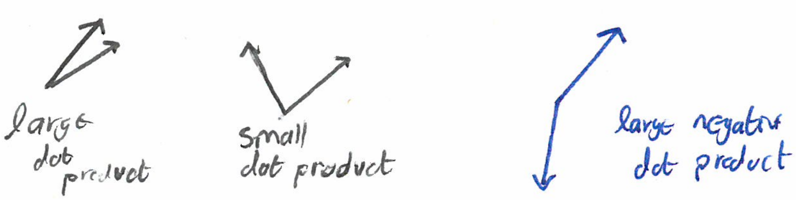
Note: $\vec{x} \perp \vec{y}$ if and only if $\vec{x} \cdot \vec{y} = 0$

is perpendicular

Dot products are "large" when θ near 0

— — — "large" and negative when θ near $-\pi$

Dot product is a measure of similarity of vectors.



Careful:

Dot product of two vectors is a scalar

Can not add a scalar and a vector

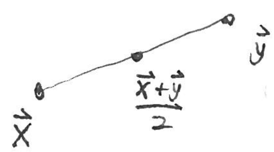
Eg $(\vec{x} \cdot \vec{y}) + \vec{z}$ doesn't make sense

Elementary functions take scalars

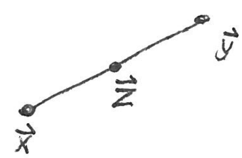
$e^{\vec{x}}$ and $\log \vec{x}$ do not make sense

Midpoint

The midpoint of $\vec{x}$ and $\vec{y}$ is $\dfrac{\vec{x}+\vec{y}}{2}$.



Pf: Let midpoint be $\vec{z}$

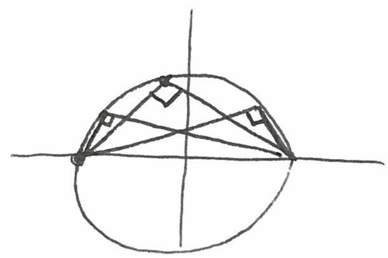


Because $\vec{z}$ is midpoint, $\vec{z}-\vec{x}=\frac{1}{2}(\vec{y}-\vec{x})$

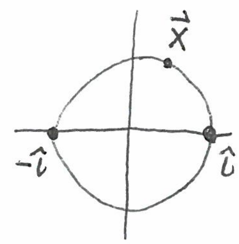
So $\vec{z}=\frac{1}{2}\vec{y}+\frac{1}{2}\vec{x}$ ∎

Geometric Proofs

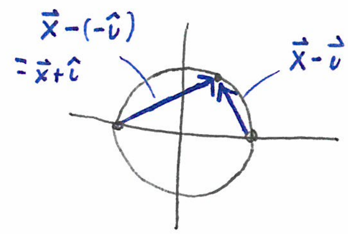
Example: Show that lines connecting any point on the unit ~~semi~~circle to $(1,0)$ and $(-1,0)$ are perpendicular



Step 1: Label important points



Step 2: Identify important vectors



Step 3: Interpret givens and goal in terms of vectors

Given: $|\vec{x}| = 1$

Goal: $\vec{x}+\hat{\imath} \perp \vec{x}-\hat{\imath}$

Step 4: Connect givens and goal by algebra

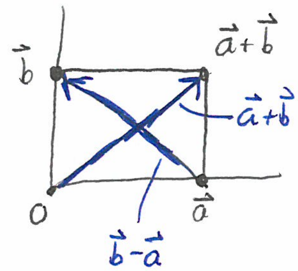
Want to show $(\vec{x}+\hat{\imath})\cdot(\vec{x}-\hat{\imath}) = 0$.

Computing,

$$\begin{aligned}(\vec{x}+\hat{\imath})\cdot(\vec{x}-\hat{\imath}) &= \vec{x}\cdot\vec{x} - \cancel{\vec{x}\cdot\hat{\imath}} + \cancel{\hat{\imath}\cdot\vec{x}} - \hat{\imath}\cdot\hat{\imath} \\ &= |\vec{x}|^2 - 1 \\ &= 1 - 1 = 0.\end{aligned}$$

Example : Show that a rectangle is a square if its diagonals are perpendicular

Label points & vectors



Given: $\vec{a} \perp \vec{b}$ and $\vec{a}+\vec{b} \perp \vec{b}-\vec{a}$

Goal: $|\vec{a}| = |\vec{b}|$

Algebra: $(\vec{a}+\vec{b}) \cdot (\vec{b}-\vec{a}) = 0$

$$\Rightarrow \cancel{\vec{a}\cdot\vec{b}} - \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} \cancel{- \vec{b}\cdot\vec{a}} = 0$$

$$\Rightarrow \vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{a}$$

$$\Rightarrow |\vec{b}|^2 = |\vec{a}|^2$$

$$\Rightarrow |\vec{b}| = |\vec{a}| \quad \blacksquare$$