Lecture 11 - Vector fields, line integrals, conservative vector fields - 7/28/2014 — Interphase 2014 Calc 3

29. Vector fields

    a. A vector field is a vector-valued function: $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.

30. Line integrals

    a. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the line integral of $\mathbf{F}$ over the curve C.

    b. A line integral is approximated by a Riemann sum

$$\int_C \mathbf{F} \cdot d\mathbf{r} \approx \sum \mathbf{F} \cdot \Delta\mathbf{r},$$

    where the curve is broken up into many tiny straight segments.

    c. The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the work done by the force field $\mathbf{F}$ on an object traveling along path C.

    d. The work done by $\mathbf{F}$ is also equal to the integral with respect to arc length of the tangential component of the force:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds.$$

    Here, $\mathbf{T}$ is the unit tangent vector to the curve and s is arc length.

    e. If the curve C is parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t)) \cdot \frac{d\mathbf{r}(t)}{dt}\, dt.$$

31. Conservative vector fields

    a. The vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is *conservative* if $\mathbf{F}(x, y) = \nabla\phi(x, y)$ for some function $\phi(x, y)$.

    b. The function ϕ is called a *potential function*. It is defined up to an additive constant.

    c. $\mathbf{F}$ is conservative if and only if $\partial_y P = \partial_x Q$.

    d. To find the potential function ϕ from $\mathbf{F}$, integrate each component of $\mathbf{F}$ and combine.