Lecture 11

28 july 2014

Vector fields

Line integrals

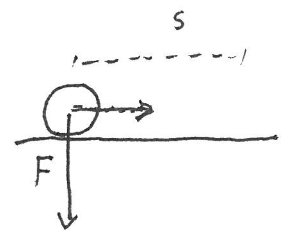
Conservative vector fields

Warmup:

a) Force due to gravity F acts on object for distance s. How much work was done on object?

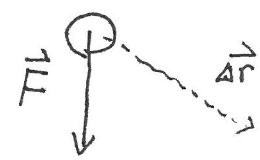
$F \cdot s$

b) Ball rolls distance s on table. What is work due to gravity?



0

c) Gravity force $\vec{F}$. object moves displacement $\Delta\vec{r}$, what is work done on object



$\vec{F} \cdot \vec{\Delta r}$

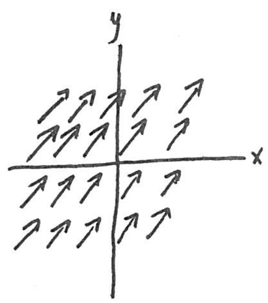
Vector Fields

A vector field is a vector-valued function

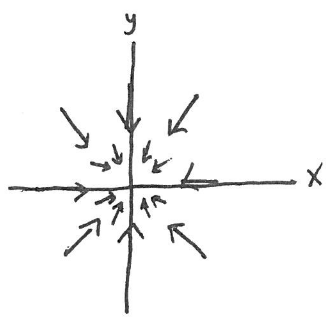
(2D) $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$

(3D) $\vec{F}(x,y,z) = P(x,y,z)\,\vec{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}$

Examples:

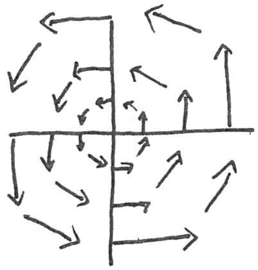


$\vec{F}(x,y) = \hat{\imath} + \hat{\jmath}$



$\vec{F}(x,y) = -x\,\hat{\imath} - y\,\hat{\jmath}$

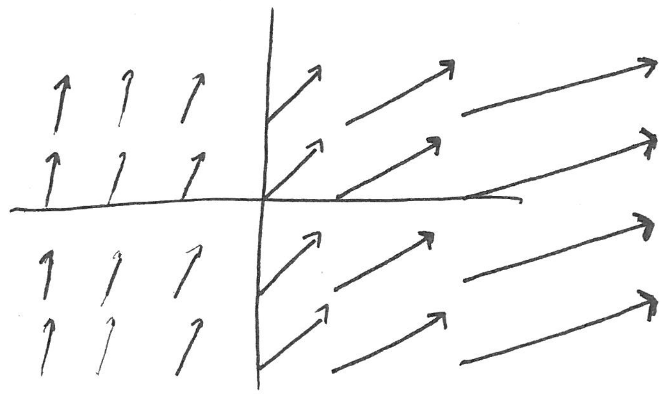
$\vec{F}(x,y) = -y\,\hat{\imath} + x\,\hat{\jmath}$



Example: Plot: $\vec{F}(x,y) = e^x \hat{i} + \hat{j}$

Note: $\hat{j}$ component always 1

$\hat{i}$ comp is positive, small for neg value of x



Plot

Example: $\vec{F} = \dfrac{(x-1)\,\hat{\imath} + (y+1)\,\hat{\jmath}}{\sqrt{(x-1)^2+(y+1)^2}}$

Denominator looks like a normalization.

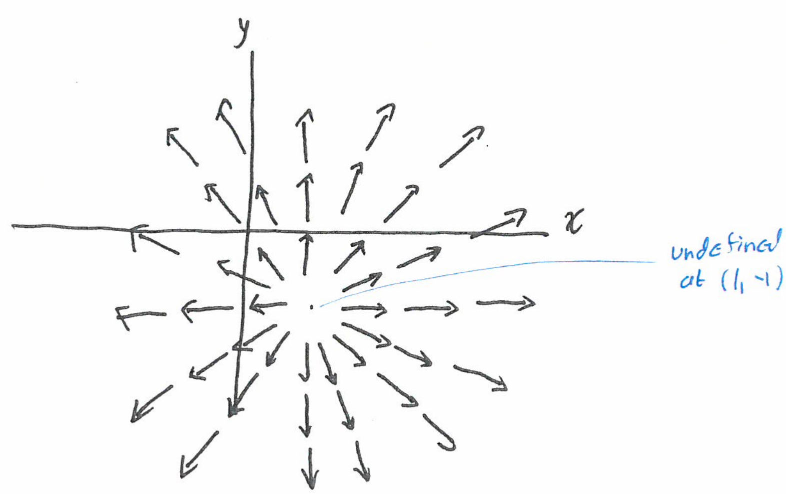
$$\sqrt{(x-1)^2+(y+1)^2} = \left|(x-1)\hat{\imath} + (y+1)\hat{\jmath}\right|$$

So $\vec{F}$ is always a unit vector.

To get a sense of direction, ignore denominator

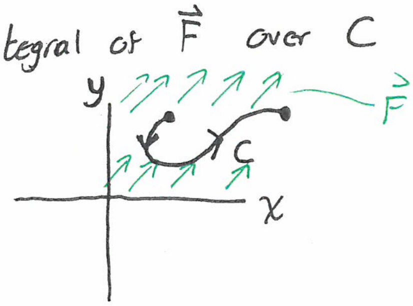
$$(x-1)\hat{\imath} + (y+1)\hat{\jmath} = \langle x, y\rangle - \langle 1, -1\rangle$$

is vector from $\langle 1, -1\rangle$ to $\langle x, y\rangle$



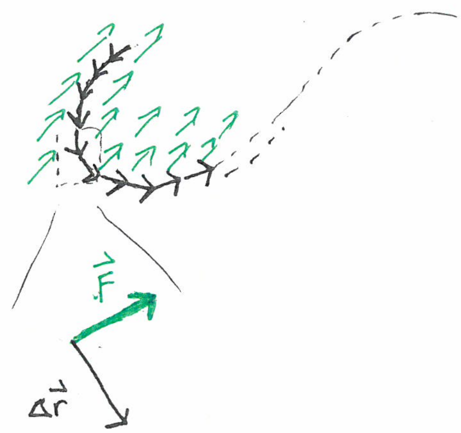
Line Integral

$\int_C \vec{F}(x,y)\cdot d\vec{r}$ is line integral of $\vec{F}$ over C



is like Riemann sum

$\sum \vec{F}(x,y)\cdot\Delta\vec{r}$



If $\vec{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$

write $\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C \langle P,Q\rangle\cdot\langle dx,dy\rangle$

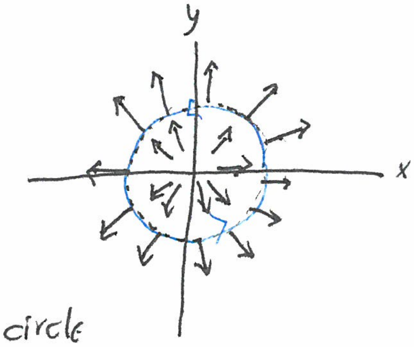
$= \int_C \vec{F}\cdot d\vec{r}$

$$\int_C \vec{F}(x,y)\cdot d\vec{r} = \int \vec{F}\cdot\underbrace{\vec{T}}_{\text{unit tangent vector}}\,\underbrace{ds}_{\text{arc length}}$$

Integral with respect to arc length of tangential component of the force

Activity:

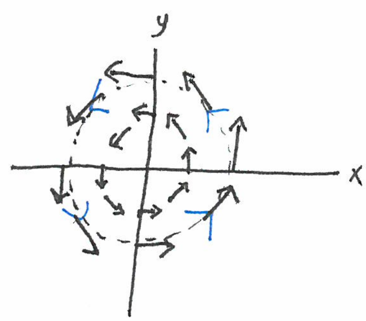
a) $\vec{F}$ is



Let c be circle
traversal counter clock wise.
Is $\int_C \vec{F} \cdot d\vec{r}$ positive, negative, or 0

b) $\vec{F}$ is

C is circle.

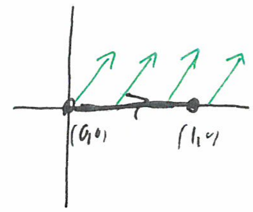


Is $\int_C \vec{F} \cdot d\vec{r}$
pos/neg/zero?

Example: $F = \hat{i} + \hat{j}$

C = line segment from $(0,0)$ to $(1,0)$

$d\vec{r}$ always in direction of $\hat{i}$



So $\int \vec{F} \cdot dr \sim (\hat{i}+\hat{j}) \cdot \hat{i} \cdot$ length of curve $= 1$

Evaluating line integral directly

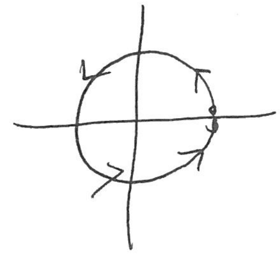
$$\int_C \vec{F} \cdot d\vec{r}$$

Parameterize by $\vec{r}(t) = \langle x(t), y(t) \rangle$ $\quad a \le t \le b$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t)) \cdot \frac{d\vec{r}}{dt}(t) \; dt$$

Doesnt depend on parameterization. All give same answer

Example: $\vec{F} = -y\hat{i} + x\hat{j}$



C = Unit circle traversal counterclockwise (positive direction)

$\vec{r}(\theta) = \langle \cos\theta, \sin\theta \rangle \quad 0 \le \theta \le 2\pi$

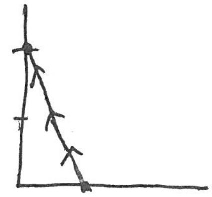
$\dfrac{d\vec{r}}{d\theta} = \langle -\sin\theta, \cos\theta \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \sin\theta, \cos\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle \, d\theta$$

$$= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta) \, d\theta = 2\pi.$$

Example: $\vec{F}(x,y) = x\hat{\imath} + y\hat{\jmath}$

C = line segment from $(1,0)$ to $(0,2)$



$\vec{r}(t) = \langle 1,0\rangle + t\langle -1,2\rangle$ for $0 \le t \le 1$

$$\frac{d\vec{r}}{dt} = \langle -1,2\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle 1-t, 2t\rangle \cdot \langle -1,2\rangle \, dt$$

$$= \int_0^1 (-1+t+4t)\,dt = \int_0^1 -1+5t \, dt$$

$$= -1 + \frac{5}{2} = \boxed{3/2}$$

Conservative Vector Field

- $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$ is conservative
  if $\vec{F}(x,y) = \nabla\varphi(x,y)$ for some $\varphi(x,y)$.

- $\vec{F}$ is conservative iff $\partial_y P(x,y) = \partial_x Q(x,y)$.

  Why? $\vec{F} = \nabla\phi \Rightarrow \vec{F} = \underbrace{\partial_x \phi}_{P}\,\hat{\imath} + \underbrace{\partial_y \phi}_{Q}\,\hat{\jmath}$

  Recall $\phi_{xy} = \phi_{yx}$. So $\partial_y P = \partial_x Q$.

- To find φ from $\vec{F}$, integrate each component and combine

- Note φ defined up to a constant

Example: $\vec{F} = \underbrace{x}_{P}\,\hat{\imath} + \underbrace{y}_{Q}\,\hat{\jmath}$. Is it conservative? Find φ

$\partial_y P = \partial_x Q = 0 \Rightarrow$ Conservative

$\partial_x \phi = x \Rightarrow \phi = \frac{1}{2}x^2 + f(y)$

$\partial_y \phi = y \Rightarrow \phi = \frac{1}{2}y^2 + g(x)$

combine

$\phi = \frac{1}{2}x^2 + \frac{1}{2}y^2$ is a potential function for $\vec{F}$

Example $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath}$ Is it conservative

$\partial_y P = -1$

$\partial_x Q = 1$

Not equal. Not conservative.

Example: $\vec{F} = (x+y)\hat{\imath} + (y+x)\hat{\jmath}$. Is $\vec{F}$ conservative? Find potential functions

$\partial_y P = 1$
$\partial_x Q = 1$ $\Rightarrow$ Conservative

To find φ:

$\partial_x \varphi = x+y \Rightarrow \varphi = \frac{1}{2}x^2 + xy + f(y)$

$\partial_y \varphi = x+y \Rightarrow \varphi = xy + \frac{1}{2}y^2 + g(x)$

Combining, $\varphi = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2$